Chapter 3

AN APPLICATION-ORIENTED THEORY OF MOSD

3.2 Multiobjective formulation of a design problem

In general, problems arising in electromagnetic design can be formulated as non-linear constrained problems. Often, multiple objective functions are to be optimized simultaneously: problems of this kind belong to the category of multiobjective or multi-criteria. Their formulation is characterized by a vector of objective functions.

Formally, considering n_v variables, a multiobjective problem can be cast as follows:

$$\text{given } x_0 \in \Re^{n_v} \text{ , find } \inf_x F(x) \text{ , } x \in \Re^{n_v} \tag{3.1}$$

subject to n_i inequality and n_e equality constraints

$$g_i(x) \leq 0 \text{ , } i = 1, n_i \tag{3.2}$$

$$h_j(x) = 0 \text{ , } j = 1, n_e \tag{3.3}$$

and also to $n_b \leq 2n_v$ bounds

$$\ell_k \leq x_k \leq u_k \text{ , } k = 1, n_v \tag{3.4}$$

It can be noted that a subset of design variables x might not belong to $\Re^{n_v}$, like in the case of discrete-valued or integer-valued variables; here, however, continuous-valued variables are considered.

In (3.1), $F(x) = \{f_1(x), ..., f_{n_f}(x)\} \subset \Re^{n_f}$ is the objective vector composed of $n_f \geq 2$ terms. Therefore, F defines a transformation from the design space $\Re^{n_v}$ to the corresponding objective space $\Re^{n_f}$. Often, the n_f objectives are non-commensurable, because they have different physical dimensions: they might refer to various characteristics or performances of the device (*e.g.* cost of materials, device volume, field homogeneity, power loss and so forth), to be optimized simultaneously. Therefore, the designer is forced to

look for best compromises among all the objectives. In order problem (3.1) to be non-trivial, the pair (f_i,f_j) must represent conflicting objectives for $i \neq j$; a rigorous definition of conflict will be given in Section 3.3 (see Definition 3.8).
Traditionally, the multiobjective problem is reduced to a single-objective one by introducing a preference function $\psi(x)$, *e.g.* the weighted sum of the objectives:

$$\psi(x) = \sum_{i=1}^{n_f} c_i f_i(x) \tag{3.5}$$

with $0 < c_i < 1$, $\sum_{i=1}^{n_f} c_i = 1$, to be minimised with respect to x. It is clear that the hierarchy attributed to one objective can be modified by changing the corresponding weight. For a given set of weights, the relevant solution, if any, is assumed to be the optimum.
However, the most general solution is represented by the Pareto front of non-dominated solutions, *i.e.* those for which the decrease of a function is not possible without the simultaneous increase of at least one of the other functions. This means to have a family of solutions to be compared. From the designer viewpoint, when multiple solutions of a given problem are available, it is necessary to rank them according to a general rule: the concept of Paretian optimality is particularly useful, because it gives a mathematically precise definition of compromise solution. The theory of multiobjective shape design (MOSD) presented in the book is based just on Paretian optimality. It is intended to be application-oriented because a problem-solving approach shall be followed: in particular, static optimization will be introduced first, while extension to dynamic optimization will be developed next.

3.3 Paretian optimality

The basic concepts featuring the Pareto optimality theory are here presented. The first definition refers to the search space, which is defined as in most minimisation problems.

Definition 3.1

Let $x \in \Re^{n_v}$ denote the n_v-dimensional design vector, *i.e.* let its components $x_k \in \Re$, $k = 1, n_v$ be the design variables.

Let $\Omega_k \subseteq \Re$, $k = 1, n_v$ be n_v one-dimensional sets, named bounding ranges, and let $g_i(x): \Re^{n_v} \to \Re$, $i = 1, n_i$ and $h_j(x): \Re^{n_v} \to \Re$, $j = 1, n_e$ be $n_i + n_e$ scalar functions, termed constraints.

Then, the set

$$X = \{x \mid x_k \in \Omega_k \ , \ k = 1, n_v \ , \ g_i(x) \le 0 \ , \ i = 1, n_i \ , \ h_j(x) = 0 \ , \ j = 1, n_e\} \tag{3.6}$$

is called feasible design region or design space.

In other words, X is the set containing all and only the design vectors fulfilling bounds and constraints. The definition of the objective space is then straightforward.

Definition 3.2

Let $X \subseteq \Re^{n_v}$ be a design space and let $F(x): X \to \Re^{n_f}$ be a vector of scalar functions $f_j(x)$, $j = 1, n_f$ termed objectives. The latter are supposed to be bounded, *i.e.* it is assumed that n_f constants $m_i \in \Re^+$ exist such that $|f_i(x)| \le m_i$, $i = 1, n_f$. Then, the set

$$Y = F(X) = \{y \in \Re^{n_f} \mid \exists x \in X \ \text{such that} \ y = F(x)\} \tag{3.7}$$

is called objective space (see Fig. 3.1).

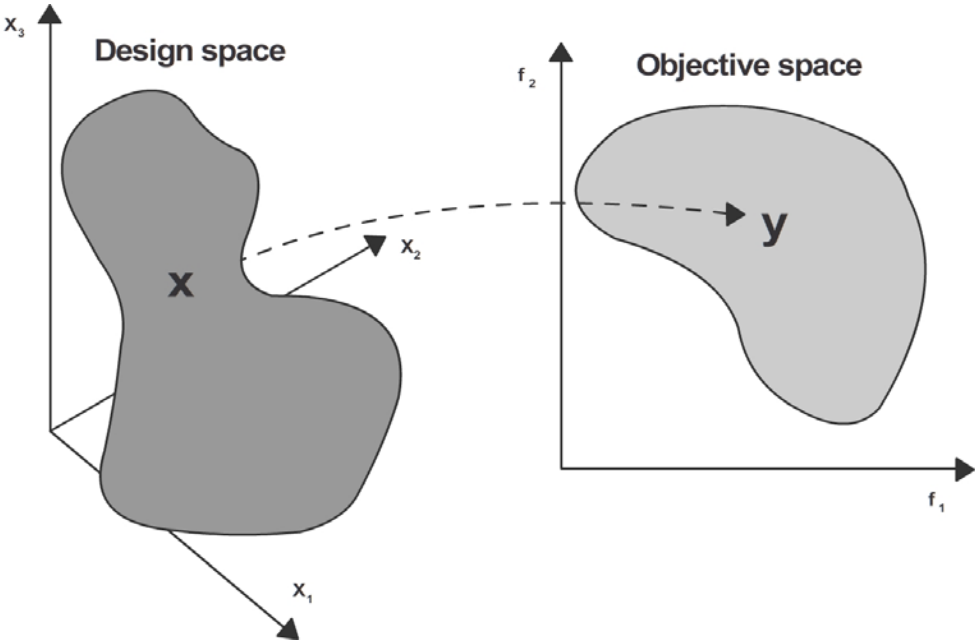

Fig. 3.1 – Mapping from design to objective space.

In general, some objectives are to be minimised and some others to be maximised. According to Paretian optimality, given two feasible solutions, a solution dominates the other one if the first is better than the second with respect to one objective, without worsening all the other objectives. On the other hand, two solutions are indifferent to each other if the first is better than the second for some objectives, while the second is better than the first in all the other objectives. This pair of concepts represents the key point of the theory, which is formalized as follows.

<u>Definition 3.3</u>

Let $X \subseteq \Re^{n_v}$ be a design space and let $F(x): X \rightarrow Y \subseteq \Re^{n_f}$ be a vector of n_f objectives, where each $f_j(x)$, $j = 1, n_f$ is to be minimised with respect to x. Given two vectors $x_1 \in X$ and $x_2 \in X$ with $x_1 \neq x_2$, the following relationships hold:

- x_1 is said to dominate x_2 if
  $\exists i$ such that $f_i(x_1) < f_j(x_2)$ and $f_j(x_1) \leq f_j(x_2)$ $\forall j = 1, n_f$, $j \neq i$;
- x_1 is said to be indifferent to x_2 if
  $\exists i$ such that $f_i(x_1) < f_j(x_2)$ and $\exists q$ such that $f_q(x_2) < f_j(x_1)$ $\forall j = 1, n_f$.

Actually, the definition above deals with the weak dominance; the definition of strong dominance follows:

- x_1 is said to strongly dominate x_2 if
  $\exists i$ such that $f_i(x_1) < f_j(x_2)$ and $f_j(x_1) < f_j(x_2)$ $\forall j = 1, n_f$, $j \neq i$.

The following remarks can be put forward.
When two solutions are available, three kinds of logically different situations can happen as shown in Table 3.1.

Table 3.1 – Solution comparison: a logical sorting.

Situation	Consequence
x_1 dominates x_2	x_1 is better than x_2
x_2 dominates x_1	x_2 is better than x_1
none of the two	x_1 and x_2 are equivalent

It should be noted that the concept of indifference does not apply under the frame of single-objective . In fact, given an objective $\psi(x)$ and two feasible vectors x_1 and x_2 with $x_1 \neq x_2$, if $\psi(x_1) \neq \psi(x_2)$, either $\psi(x_1) < \psi(x_2)$ or $\psi(x_2) < \psi(x_1)$ holds.

Moreover, in the case of n_f = 2 objectives, there is a straightforward geometric representation of dominance relationships. Any point of the Y space, whose coordinates take the values of the two objectives, can be located at the vertex of the dominance dihedral. The latter is defined as the orthogonal sector having its vertex at a given point in the Y space, and containing all and only the points dominating the given one (see Fig. 3.2) according to Def. 3.3. If the dihedral is empty, then the solution corresponding to the vertex is said to be non-dominated. An analogous definition of hyper-dihedral can be given for a number n_f > 2 of objectives.

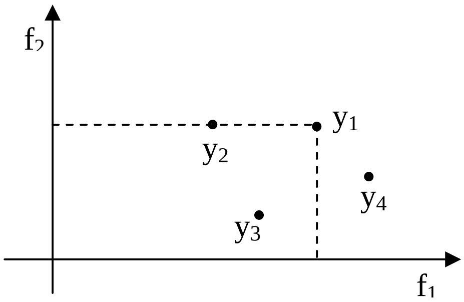

Fig. 3.2 – Geometric interpretation of dominance dihedral (n_f = 2).

For the sake of an example, supposing a min-min problem, the situation depicted in Fig. 3.1 is the following: given $F: X \rightarrow \Re^2$, $F^{-1}(y_2)$ (weakly) dominates $F^{-1}(y_1)$; $F^{-1}(y_3)$ (strongly) dominates $F^{-1}(y_1)$; $F^{-1}(y_4)$ is indifferent to $F^{-1}(y_1)$.

It is intuitive to realize that the ideal goal of a multiobjective is to find all and only the non-dominated solutions, *i.e.* those which are not dominated by any other in the design space or, alternatively, those the dominance dihedral of which is empty (see Fig. 3.3).

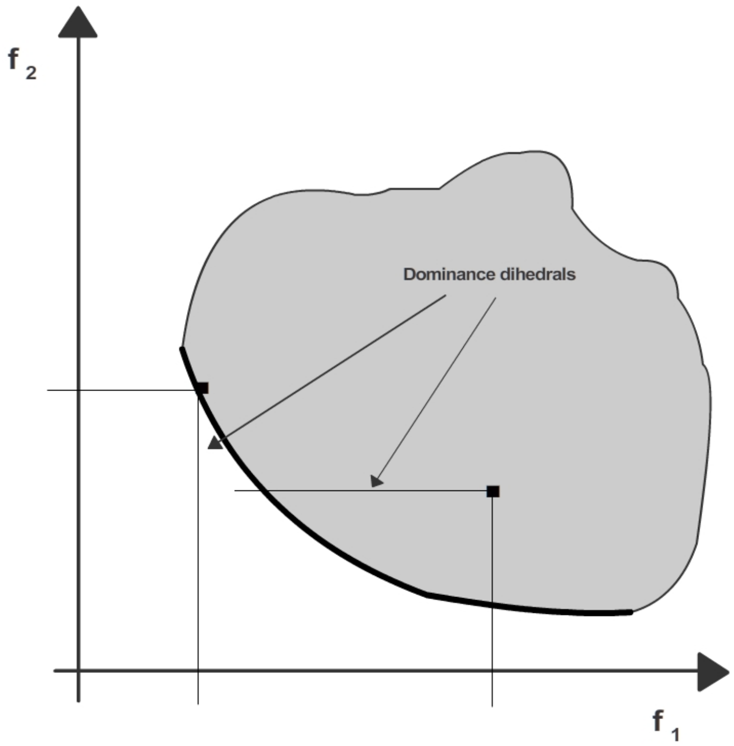

Fig. 3.3 – An objective space bounded by the PF:
the dominance dihedral of a solution along the front is empty.

Non-dominated solutions are called also non-inferior or efficient solutions. So, the definition of optimality follows immediately.

Definition 3.4

Let $Y \subseteq \Re^{n_f}$ be an objective space. Then, a point $y \in Y$ is said to be Pareto optimal if no point $\tilde{y} \in Y$ exists such that $F^{-1}(\tilde{y})$ dominates $F^{-1}(y)$.

The next definition is preliminary to the formulation of a multiobjective problem.

Definition 3.5

Let $F(x): X \to Y$ be a vector of n_f objectives, with $X \subseteq \Re^{n_v}$ and $Y \subseteq \Re^{n_f}$ indicating the design space and the objective space, respectively. Then,

- the set $\Phi = \{ y \in Y \mid y \text{ is Pareto optimal} \}$ is called Pareto front (PF);
- the set $\Xi = \{ x \in X \mid F(x) \in \Phi \}$ is called Pareto set (PS).

Sometimes, the Pareto front Φ is called trade-off curve.

The sets Ξ and Φ characterise the solution of a multiobjective problem in the X and Y spaces, respectively; in particular, Φ is the image of Ξ through F. In a sense, Ξ should be considered the solution set, in analogy to the single-objective case (see Fig. 3.4)

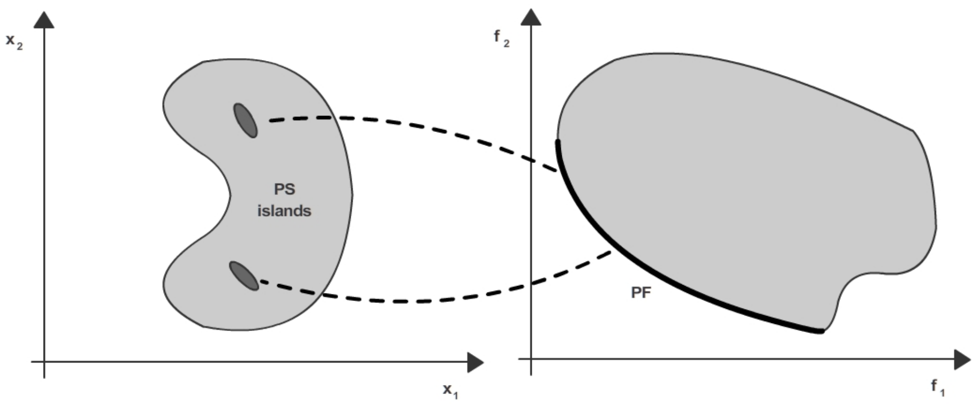

Fig. 3.4 – Correspondence between PF and PS:
the latter might form a set of islands.

In particular, in the design space it might happen that non-dominated solutions fall inside a set of sub-regions (PS islands), a kind of statistical attractor, which however map to a small part of the front: getting information about the structure of the search space helps to select

appropriate solution methods for the given optimisation problem (see Chapter 5).
According to previous definitions, it is possible to give a general formulation for a problem with any number of objectives.

Definition 3.6
Let (F, X, Y) be the ensemble of a vector objective function with the relevant design and objective spaces, respectively. Then, the corresponding multiobjective minimisation problem reads

$$\text{find } \Phi \subset Y \text{ and } \Xi \subset X \text{ such that } F(\Xi) = \Phi \qquad (3.8)$$

The last definition, which generalizes the formulation (3.1)-(3.4) given in Section 3.2, points out the fact that a multiobjective problem admits a set of solutions, instead of just one like in the single-objective case. This represents the peculiarity of the theory presented.
Another remark is worth being made. In an engineering problem, what is really interesting is to find the set of values $\Xi = F^{-1}(\Phi)$ which minimises the objective functions in the Pareto sense. The objective space Y is but the control space, *i.e.* it provides some metrics to identify non-dominated solutions, which in turn are given in terms of the design space X.
Following this line, a preliminary information about how the objective space is bounded is provided by some characteristic points, which can be determined prior to solving the multiobjective problem. Referring to a min-min problem, as each scalar objective function is bounded and so it is supposed to have a minimum, there exists a vector in the Y space, called ideal objective vector, whose coordinates are the minima of the corresponding single objectives.

Definition 3.7
Let (F, X, Y) with $X \subseteq \Re^{n_v}$ and $Y \subseteq \Re^{n_f}$ represent a multiobjective problem. Then, the point

$$U = \left(U_1, ..., U_i, ..., U_{n_f}\right) \text{ with } U_i = \inf_x f_i(x) \text{ , } i = 1, n_f \qquad (3.9)$$

is the ideal objective vector, named utopia point.

The denomination of utopia descends appropriately from the impossibility to obtain such a solution in any way. In other words, it does not exist a vector $\tilde{x} \in X$ such that $F(\tilde{x}) = U$. On the other hand, if such a vector existed, it would be the unique solution to the problem; however, the latter would be single-objective instead of multiobjective.

In turn, the definition of utopia gives a chance to formalize the idea of conflict among multiple objectives. The following definition holds.

Definition 3.8

Given n_f objectives, they are said to be in conflict if $\exists \tilde{x}_i \in X$ such that

$$f_i(\tilde{x}_i) = \inf_x f_i(x) \tag{3.10}$$

with $\tilde{x}_i \neq \tilde{x}_j$, $i \neq j$, $i, j = 1, n_f$.

The conflict among objectives is just what prevents from minimising all of them simultaneously.

In a symmetrical way with respect to utopia, the anti-utopia point can be defined.

Definition 3.9

Let (F, X, Y) with $X \subseteq \Re^{n_v}$ and $Y \subseteq \Re^{n_f}$ represent a multiobjective problem. Then, the point

$$A = \left(A_1, ..., A_i, ..., A_{n_f}\right) \text{ with } A_i = \sup_x f_i(x) \text{ , } i = 1, n_f \tag{3.11}$$

is called anti-utopia point.

Finally, the definition of the nadir point is possible, even if it is univocal only in the case of two objectives.

Definition 3.10

Let (F, X, Y) with $X \subseteq \Re^{n_v}$ and $Y \subseteq \Re^{n_f}$ represent a two-objective problem, *i.e.* $n_f = 2$. Then, the point

$$R = (R_1, R_2) \text{ with } R_j = \sup_{\Phi} y_j \text{ , } j = 1,2 \tag{3.12}$$

is called nadir point.

Extending the definition of nadir to the case $n_f > 2$ is not straightforward, for the dimensionality of the Φ set grows correspondingly and so the number of its end points. To this end, a wider information on the PF can be obtained after evaluating the matrix M with entries defined as follows:

$$\begin{aligned} M_{ij} &= U_i \text{ , } i = j \\ M_{ij} &= f_j(x)\big|_{f_i(x)=U_i} \text{ , } i \neq j \end{aligned} \tag{3.13}$$

with indexes i and j ranging from 1 to n_f . In terms of just matrix M, the nadir coordinates can be obtained as follows:

$$R_i = \max_{j=1:n_f} M_{ij} \;,\; i = 1, n_f \qquad (3.14)$$

From a practical point of view, the computation of the $n_f \times n_f$ matrix M, and so of vectors U and R, requires n_f single-objective optimisations.

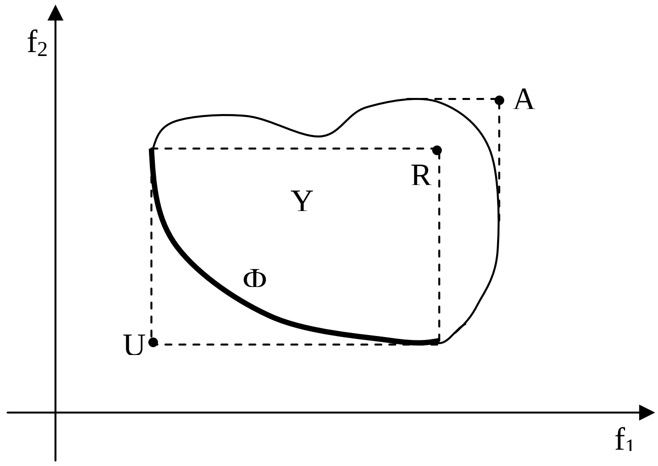

Fig. 3.5 – Geometric interpretation of the objective space (n_f = 2).

The overall situation is represented in Fig. 3.5 for the two-dimensional case; it can be remarked that the representation of the front is straightforward for two objectives, difficult for three objectives, impossible for more than three objectives.
Basically, the pairs of points (U,R) and (U,A) provide two different bounds to Φ; moreover, it can be noted that U and A are unfeasible points by definition, while point R might be feasible. These points provide also a metric criterion: in fact, the distance between two points belonging to the Y space can be measured taking $\|R - U\|$ as the reference distance.
A final remark can be put forward: Φ can be interpreted as the boundary between the feasible design region Y and the unfeasible one outside Y, within the dominance dihedral associated to point R.

The geometric classification of the front is far more than a theoretical aspect; actually, if priorly known, it can be crucial in choosing the proper method to identify the front, or, conversely, in determining the failure of an inadequate one.

More formally, the PF can be written as a function of one out of n_f objectives against all the others in the following way:

$$\tilde{f}_{n_f} = \tilde{f}_{n_f}\left(f_1,\dots,f_{n_f-1}\right) \qquad (3.15)$$

where $\tilde{f}_{n_f}$ denotes the non-dominated values of the n_f-th objective. The latter function can be expressed in a closed form only in few analytical cases.

Definition 3.11

A multiobjective is said to be convex if and only if all objective functions and all constraint functions are convex; conversely, it is non-convex if at least one objective or constraint is non-convex.

It could be proven that, for a convex problem, function (3.15) is convex and *vice versa*.

Definition 3.12

A front is said to be discontinuous - or non-connected - if and only if function (3.15) is discontinuous.

If this happens, the front is composed of a number of non-connected branches.

Definition 3.13

In a two-dimensional case, a front is said to be weakly Paretian if either f_1 or f_2 , or both of them, are constant along a part of it.

In Fig. 3.6 four main topologies of PF are shown for a two-dimensional min-min problem: convex, non-convex, non-connected, weakly Paretian, respectively.

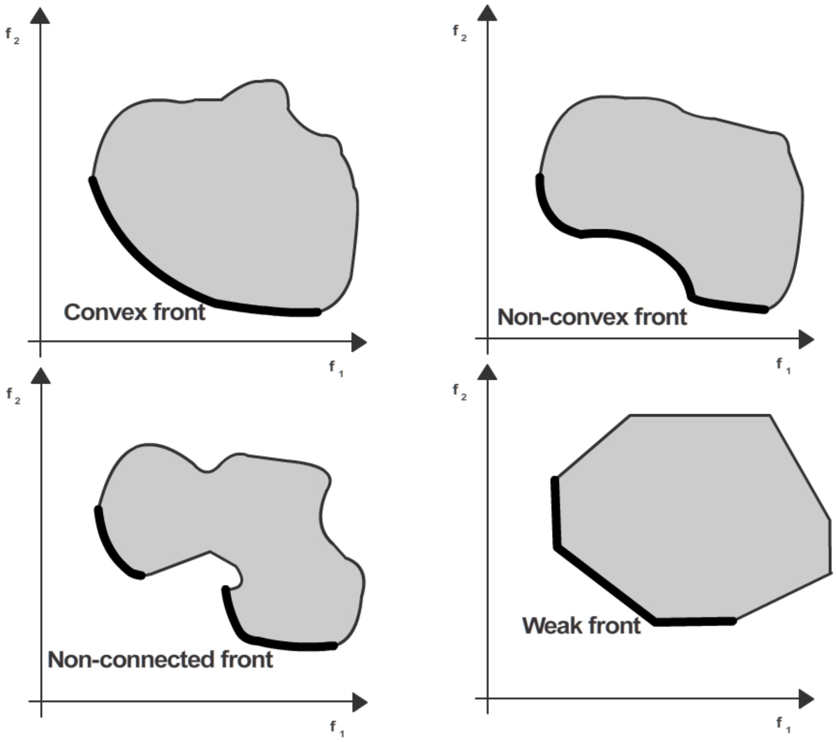

Fig. 3.6 – Four possible topologies of PF.

Nevertheless, there is another remark about front classification which relates to the difficulty of sampling it satisfactorily. Due to non-linearity of objective functions, in general, sampling uniformly the design space X gives rise to non-uniformly spaced solutions on the front in Y or, from the reverse viewpoint, if uniform sampling of the front is wanted in Y, sampling in X must follow a non-uniform law which is *a priori* unknown (see Fig. 3.7).
If S_X is a random sampling of space X, let $S_Y=F(S_X)$ be the corresponding sampling of space Y through vector transformation F. The concept of deceptive front arises.

<u>Definition 3.14</u>
A front is said to be deceptive if and only if S_Y is a non-uniform distribution when S_X is a uniform distribution.

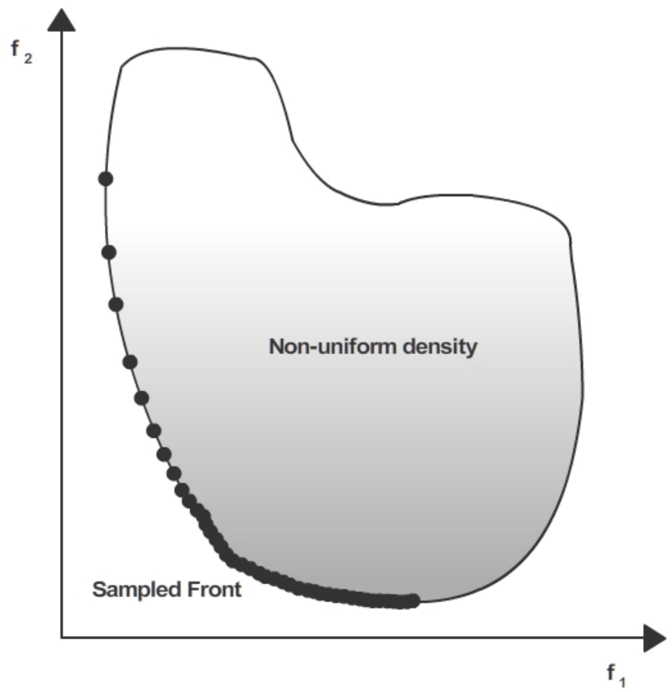

Fig. 3.7 – Non-uniformly sampled PF (deceptive topology).

Finally, the extension of the concept of multiple minima arising in single-objective to the multi-objective case gives origin to the definition of multimodal problem.

Definition 3.15
A multiobjective problem is said to be multimodal if and only if more than one PF exists.
In general, the presence of one or more local fronts in addition to the global front is due to the non-convexity of at least one out of n_f objective functions.

For the sake of an example, a simple analytical benchmark is considered first:

$$\text{find} \inf_{(x_1,x_2)} (f_1, f_2) \quad , \quad (x_1, x_2) \in \Re^2 \tag{3.16}$$

with

$$f_1(x_1, x_2) = x_1^2 + x_2^2 \tag{3.17}$$

$$f_2(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2 \tag{3.18}$$

Due to the symmetry of f_1 and f_2 in the design space, the equation of the Pareto set is

$$x_1 = x_2 \quad , \quad (x_1, x_2) \in [0,1] \times [0,1] \tag{3.19}$$

i.e. the segment of straight line joining the minima (0,0) and (1,1) of the two objectives. From (3.18)-(3.20) the equation of the PF in the objective space is

$$f_2 - f_1 + 2\sqrt{2}\sqrt{f_1} - 2 = 0 \quad , \quad f_1 \in [0,2] \tag{3.20}$$

For both functions are convex, the PF is convex too. The whole objective space is given by

$$f_2 \geq f_1 - 2\sqrt{2}\sqrt{f_1} + 2 \quad ; \quad f_i \geq 0 \quad , \quad i = 1,2 \tag{3.21}$$

In the single-objective case, a non-convex function exhibits one or more local minima in addition to the global minimum. Likewise, in the multiobjective case, the presence of non-convex functions originates local Pareto fronts.

Definition 3.16

Let $P \subset X \subseteq \Re^{n_v}$ be a set of non-dominated solutions ξ. If $\forall \xi \in P$ no point $\tilde{\xi} \in X$ exists such that $\tilde{\xi}$ dominates ξ and $\left\| \tilde{\xi} - \xi \right\| \leq \varepsilon$, where $\varepsilon > 0$ is a small length, holds, then P is a local Pareto set; the corresponding image $F(\xi)$ is a local Pareto front.

Accordingly, an analytical example can be stated as follows:
find

$$\inf_{x \in \Omega} (f_1, f_2) \quad , \quad \Omega \subset \Re^1 \tag{3.22}$$

with

$$f_1(x) = x \cos x \quad , \quad f_2(x) = x \sin x \tag{3.23}$$

If Ω is large enough, both the objectives have several local minima; correspondingly, local fronts are originated.

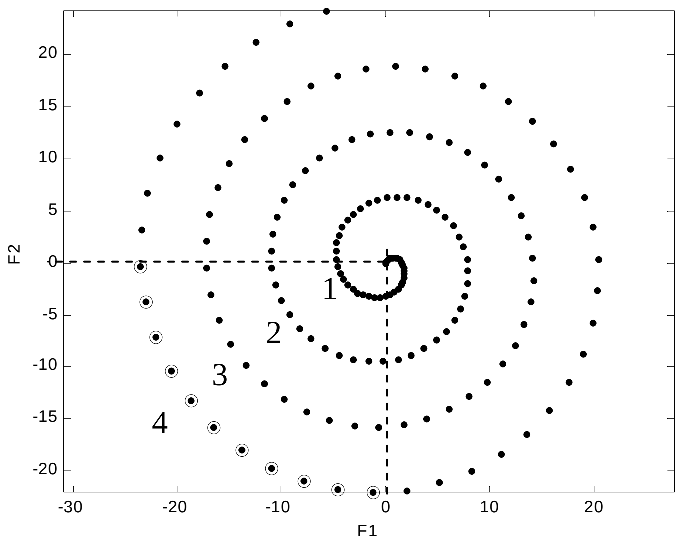

Fig. 3.8 – Multimodality: an objective space
with local (1,2,3) and global (4) fronts.

Let *e.g.* the set $\Omega = [0, 25]$ be considered; in Fig. 3.8 a discrete representation of the objective space is shown: three local fronts are outlined, in addition to the global one.
In general, the equation of the fronts can be deduced from (3.23) as

$$f_1^2 + f_2^2 = \left(\tan^{-1} \frac{f_2}{f_1} \right)^2 \; ; \; f_i < 0 \; , \; i = 1,2 \tag{3.24}$$

Moving from analytical examples to numerical case studies is necessary for applying Paretian optimality to field-based inverse problems. To this end, in the next Chapter the most important field models used in computational electromagnetism are reviewed in the light of shape design problems.