Chapter 4

SOLVING MULTIOBJECTIVE OPTIMISATION PROBLEMS

5.1 Classical methods of multiobjective optimisation

5.1.1 Objective weighting

The formulation is the following:

$$\text{find } \inf_{x} \psi(x) \ , \ x \in \Omega \subseteq R^{n_v}$$

with

$$\psi(x) = \sum_{i=1}^{n_f} \frac{w_i f_i(x)}{R_i - U_i} \ , \ \sum_{i=1}^{n_f} w_i = 1 \qquad (5.1)$$

where R_i and U_i are nadir and utopia point components, respectively. The use of this kind of normalization is important for weights w_i to express desired compromises.

The following theorem holds.

Let the PF be convex and let $F(x) = (f_1(x), f_2(x), ..., f_{n_f}(x))$. If $x^* \in \Omega$ exists such that $F^* = F(x^*) \in PF$, then there exists a weighting vector $w = \{w_i\}$, $w_i \in \Re^+$, $\sum_{i=1}^{n_f} w_i = 1$ such that x* is a solution of the problem: find $\inf_{x} w^T F(x)$, $x \in \Omega \subseteq R^{n_v}$.

A simple geometric interpretation in the two-dimensional case is shown in Fig. 5.1, where some contour lines of function $\psi = w_1 f_1 + w_2 f_2$ (dashed straight lines) are plotted in the objective space, assuming $R_i - U_i = 1$; the relevant PF is also shown.

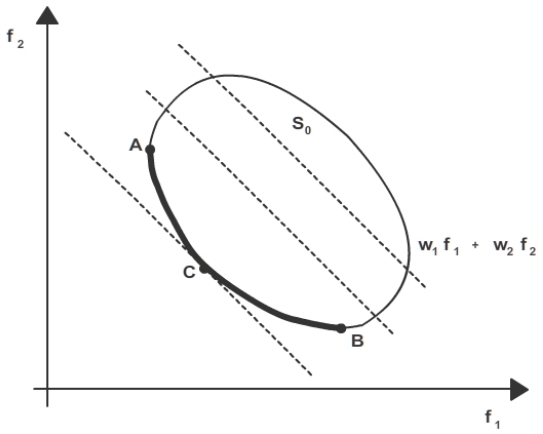

Fig. 5.1 – Success of objective weighting in the case of convex front (solution C is recovered).

Once weights w_1 and w_2 are fixed, the minimisation process can be viewed as a search for the point of the straight line which is externally tangent to the PF.
The following remarks can be put forward.

- It is easy to realize that only in the case of convex PF all its points can be sampled with this technique, by varying weights. When the front is non-convex, some solutions are missed (see Fig. 5.2).

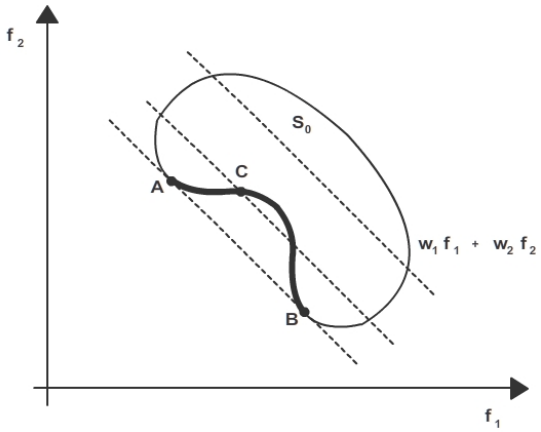

Fig. 5.2 – Failure of objective weighting in the case of non-convex front (solution C is missed).

- It is difficult to identify the distribution of weights that give rise to a uniform distribution of solutions on the PF. Though being desirable, a non-uniform distribution of weights, giving rise to a regularly spaced distributions of solutions along the PF, is *a priori* unknown.

5.1.2 Epsilon-constraint formulation

In order to remedy the difficulty of dealing with a non-convex front, a single-objective formulation is considered in which n_f-1 objectives are used as constraints and only one is selected as the objective function. The formulation reads:
given a set of n_f-1 values $\{\varepsilon_j\}$, $\varepsilon_j \in \Re$

$$\text{find } \inf_x f_i(x) \,,\ x \in \Omega \subseteq R^{n_v}$$

subject to

$$f_j(x) \le \varepsilon_j \ ,\ j \ne i \ ,\ j = 1,...,n_f \tag{5.1}$$

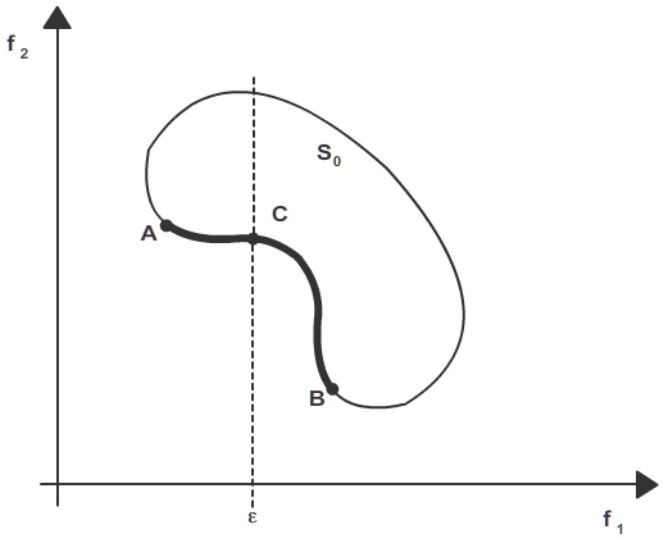

Fig. 5.3 – Success of epsilon-constraint in the case of non-convex front (solution C is recovered).

To find a Pareto-optimal solution, a suitable value ε_j is chosen for the j-th objective. Fig.5.3 shows that the method works also with a non-convex front.
The following remarks can be put forward.

- The optimisation procedure can be iterated with different values of ε_j for the j-th objective to find different Pareto-optimal solutions.
- A knowledge of an appropriate range of ε_j values for the j-th objective is required to be known *a priori*.

Chapter 5

FIELD MODELS AND SHAPE DESIGN

Direct problems considered are boundary-value problems governed by Maxwell's equations of electromagnetic field; their formulation in terms of vectors and potentials is here revisited.

4.1 Maxwell's equations in differential form

In a simply-connected domain Ω with boundary Γ, filled in by a linear material characterized by permittivity ε, permeability μ and conductivity σ, the time-varying electromagnetic field is governed by the following equations:

Faraday's equation	$\overline{\nabla} \times \overline{\mathrm{E}} = -\dfrac{\partial \overline{\mathrm{B}}}{\partial \mathrm{t}}$	(4.1)
Gauss's electric equation	$\overline{\nabla} \cdot \overline{\mathrm{D}} = \rho$	(4.2)
Ampère's equation	$\overline{\nabla} \times \overline{\mathrm{H}} = \overline{\mathrm{J}} + \dfrac{\partial \overline{\mathrm{D}}}{\partial \mathrm{t}}$	(4.3)
Gauss's magnetic equation	$\overline{\nabla} \cdot \overline{\mathrm{B}} = 0$	(4.4)

where scalar ρ is the charge density [Cm^{-3}] and vectors are defined as follows:

$\overline{\mathrm{D}}$ electric displacement [Cm^{-2}]
$\overline{\mathrm{E}}$ electric field intensity [Vm^{-1}]
$\overline{\mathrm{B}}$ magnetic induction [T]
$\overline{\mathrm{H}}$ magnetic field intensity [Am^{-1}]
$\overline{\mathrm{J}}$ current density [Am^{-2}]

In a three-dimensional domain, equations (4.1)-(4.4) are a set of eight scalar equations which the following constitutive laws

$$\overline{D} = \varepsilon \overline{E} \tag{4.5}$$

$$\overline{B} = \mu \overline{H} + \overline{B}_0 \tag{4.6}$$

$$\overline{J} = \sigma \overline{E} + \overline{J}_0 \tag{4.7}$$

must be added to; in (4.6) the term $\overline{B}_0$ accounts for the permanent magnetization of the magnetic material, if any. In (4.7) the terms $\sigma\overline{E}$ and $\overline{J}_0$ account for eddy and driving current density, respectively, while in (4.3) $\frac{\partial \overline{D}}{\partial t}$ is the displacement current density, respectively.

In total, considering both Maxwell's equations and constitutive laws, fifteen scalar unknowns (*i.e.* field components) have to be determined subject to suitable boundary and initial conditions. The system of eight plus nine equations can be solved, because there are two relations among the unknowns which are automatically satisfied.

In fact, taking the divergence of (4.1) and the time derivative of (4.4), one obtains an identity. Analogously, taking the divergence of (4.3) and the time derivative of (4.2), the charge continuity equation

$$\overline{\nabla} \cdot \overline{J} + \frac{\partial \rho}{\partial t} = 0 \tag{4.8}$$

follows. In other words, the source terms ρ and $\overline{J}$ are not independent; this makes the field analysis problem a well-posed one.

Helmholtz's theorem

A vector field $\overline{V}$ is defined in a simply-connected domain Ω, giving its divergence and curl in Ω as well as its normal component on the boundary Γ.

In a domain Ω bounded by Γ, given

$$\overline{\nabla} \cdot \overline{V} = s \qquad \text{in } \Omega \tag{2.1.16}$$

$$\overline{\nabla} \times \overline{V} = \overline{c} \tag{2.1.17}$$

$$\overline{V} \cdot \overline{n} = h \qquad \text{along } \Gamma \tag{2.1.18}$$

where $\overline{n}$ is the outward normal unit vector, the vector field $\overline{V}$ is defined in a unique way.

The following remarks can be put forward.

i) The value of s, $\overline{c}$ and h, which are the sources of the field, cannot be chosen arbitrarily.

In fact:

a) $\overline{c}$ must be divergence-free (solenoidal)

$$0 = \overline{\nabla} \cdot \left(\overline{\nabla} \times \overline{V} \right) = \overline{\nabla} \cdot \overline{c} = 0 \tag{2.1.31}$$

b)

$$\int_{\Omega} \overline{\nabla} \cdot \overline{V} \, d\Omega = \int_{\Gamma} \overline{V} \cdot \overline{n} \, d\Omega \tag{2.1.32}$$

i.e.

$$\int_{\Omega} s \, d\Omega = \int_{\Gamma} h \, d\Gamma \tag{2.1.33}$$

ii) Specifying just $\overline{\nabla} \cdot \overline{V} = s$ is not enough to determine $\overline{V}$ since also $\overline{V} + \overline{\nabla} \times \overline{W}$ fulfils (2.1.16)

$$\overline{\nabla}\cdot\left(\overline{\mathrm{V}}+\overline{\nabla}\times\overline{\mathrm{W}}\right)=\overline{\nabla}\cdot\overline{\mathrm{V}}+\overline{\nabla}\cdot\overline{\nabla}\times\overline{\mathrm{W}}=\mathrm{s} \tag{2.1.34}$$

Conversely, $\overline{\mathrm{V}}$ is not determined by just $\overline{\nabla}\times\overline{\mathrm{V}}=\overline{\mathrm{c}}$ since $\overline{\mathrm{V}}+\overline{\nabla}\Psi$ also obeys (2.1.17)

$$\overline{\nabla}\times\left(\overline{\mathrm{V}}+\overline{\nabla}\Psi\right)=\overline{\nabla}\times\overline{\mathrm{V}}+\overline{\nabla}\times\overline{\nabla}\Psi=\overline{\mathrm{c}}+0 \tag{2.1.35}$$

iii) If s = 0, the field is called solenoidal.

iv) If $\overline{\mathrm{c}}=0$, the field is called irrotational.

The problem of finding the field $\overline{\mathrm{V}}$ in a domain, knowing its sources and normal component on the boundary, is normally referred to as a *boundary-value problem*. As shown by the Helmholtz's theorem, this problem has, at most, a unique solution.

4.6 Field-based inverse problems

In multiobjective shape design, vector x represents the geometric variables of the device to be synthetised. This fact in itself makes the dependence of the objective f_j rather complex. In fact, both the direct problem, through field equations, and the inverse problem, through objectives, depend on design variables. Furthermore, since objective f_j is usually a field-related quantity, it depends on x explicitly and also implicitly, by means of the field solution. In general, the following mapping applies

$$\text{geometry } \{x\} \rightarrow \text{field } s(x) \rightarrow \text{objective } f_j(x, s(x)) \ , \ j = 1, n_f$$

Accordingly, the minimisation problem correctly reads

$$\text{find } \inf_x f_j(x, s(x)) \ , \ x \in \Omega \subseteq R^{n_v} \ , \ j = 1, n_f \tag{4.89}$$

where s(x) is the solution of the direct problem corresponding to the actual design vector x. In a problem of shape design, two aspects are always involved: the optimal synthesis of field s which takes place in the device and the optimal design of device geometry x; formulation (4.89) points out that these two aspects are tightly interconnected.

The situation is even more complicated, because $p \geq 1$ constraints might be prescribed for the field; in other words, a set

$$C = \{ s(x) \mid g_k(x, s(x)) \leq c_k \ , \ c_k \in \Re \ , \ k = 1, p \}$$

can be defined. In this case, the minimisation problem reads

$$\text{find } \inf_x f_j(x, s(x)) \ , \ x \in \Omega \subseteq R^{n_v} \ , \ j = 1, n_f \tag{4.90}$$

subject to $s(x) \in C$.

For future reference, notation $f_j(x)$ without specifications will stand for the above formulation.

Shape design problems can be classified according to the j-th objective function f_j . It can represent the discrepancy between computed and prescribed quantity or the value of a local quantity (*e.g.* a field component in a part of the device) or, more generally, some characteristics of the device, like weight or volume or cost, and so on. If, for instance, r(x) is the residual vector representing the discrepancy between desired and actual value of the quantity to be optimised, then $f_j(x) = [r(x)]^T r(x)$ (least-square formulation)

or $f_j(x) = \sup_j r_j(x)$ (min-max formulation) are possible forms of the objective.

The solution of optimisation problems like (4.89) is quite troublesome: in fact, function f_j may be neither differentiable nor convex; from the numerical viewpoint, f_j could be non-smooth a function. Moreover, the function evaluation in (4.89) or the constraint evaluation in (4.90) is costly, because any function call requires at least a solution of the field equation, which might be a non-linear one. This is the main source of insidiousness for field-based inverse problems, which calls for a trade-off between computational cost, runtime, and accuracy.

From the numerical viewpoint, the solution of optimal design problems requires, as a rule, a module for calculating the field, associated with a module performing the minimisation of an objective function.

In traditional computer-aided design (CAD) these two modules are linked in a way which simulates a trial-and-error procedure. So, starting from an initial design, the field analysis is obtained by means of a numerical method. Then, a check is made whether the device has the desired properties; if not, some variables of the model are updated and the field analysis is repeated, up to a degree of designer satisfaction. This procedure is cumbersome and time consuming.

The goal of automated optimal design (AOD), conversely, is to obtain a tight integration of the module devoted to the field analysis with the module containing the minimisation algorithm, so that the design process is fully automated. This means that the two modules are linked together in a loop. The analysis of field can be performed either by differential methods originating from Maxwell's equations (finite-difference, finite-element method), or by integral methods amenable to Green's theorems (boundary element method). In turn, numerical minimisation can be achieved by means of deterministic or evolutionary methods (see Chapter 2); the combination of any method for analysis and any method for minimisation gives origin to a variety of iterative procedures for solving an optimal design problem.

Nowadays, most of commercially available codes devoted to electromagnetic field analysis are based on the finite-element method (FEM): they proved, in fact, to offer a general-purpose and flexible tool of field simulation. In particular, commercial codes are equipped with a user interface, which enables the designer to develop a model in two or three dimensions by means of graphical operations only. These features make the simulation environment rather friendly and easy to use; so, in practice, FEM has become the most popular one, mainly in an industrial centre for research and development.

4.8 A unifying view of analysis and synthesis

In Section 4.3 it has been shown that the mathematical model of a steady-state field problem is represented by the Poisson's equation for electric or magnetic potential, subject to suitable boundary conditions. It is known that the solution of a given boundary-value problem minimises the energy functional of the field region: this corresponds to the principle of minimum energy for a physical system at steady state. The latter remark supports the alternative idea to obtain potential distribution by minimising the energy functional related to the field problem.
To this end, let the following problem be considered:

$$-\overline{\nabla}\cdot \mathrm{p}\overline{\nabla}\mathrm{u} = \mathrm{f} \quad \text{in } \Omega \tag{4.91}$$

where f is the source density, p is the material property, and u is the potential, subject to

$$\mathrm{u}=\text{const along } \Gamma_1 \tag{4.92}$$

and

$$\frac{\partial \mathrm{u}}{\partial \mathrm{n}} = 0 \ \text{ along } \Gamma_2 \tag{4.93}$$

with $\Gamma_1 \cup \Gamma_2 = \partial\Omega$. The energy functional corresponding to (4.91) is defined as follows:

$$\chi = \frac{1}{2}\int_{\Omega} \mathrm{p}\overline{\nabla}\mathrm{u}\cdot\overline{\nabla}\mathrm{u}\,\mathrm{d}\Omega - \int_{\Omega} \mathrm{fu}\,\mathrm{d}\Omega \tag{4.94}$$

where $\frac{1}{2}\mathrm{p}\overline{\nabla}\mathrm{u}\cdot\overline{\nabla}\mathrm{u}$ and fu are energy densities [Jm^{-3}]. In (4.94) the first integral term accounts for the field-matter interaction, while the second term gives the energy coming from the field source.
The first-order variation $\delta\chi$ of functional (4.94) with respect to potential u is given by:

$$\begin{aligned}
\delta\chi &= \frac{1}{2}\int_{\Omega} \mathrm{p}\,\delta\left(\overline{\nabla}\mathrm{u}\cdot\overline{\nabla}\mathrm{u}\right)\mathrm{d}\Omega - \int_{\Omega} \mathrm{f}\,\delta\mathrm{u}\,\mathrm{d}\Omega = \\
&= \frac{1}{2}\int_{\Omega}\left[\mathrm{p}\overline{\nabla}(\delta\mathrm{u})\cdot\overline{\nabla}\mathrm{u} + \mathrm{p}\overline{\nabla}\mathrm{u}\cdot\overline{\nabla}(\delta\mathrm{u})\right]\mathrm{d}\Omega - \int_{\Omega} \mathrm{f}\,\delta\mathrm{u}\,\mathrm{d}\Omega = \\
&= \int_{\Omega} \mathrm{p}\overline{\nabla}\mathrm{u}\cdot\overline{\nabla}(\delta\mathrm{u})\,\mathrm{d}\Omega - \int_{\Omega} \mathrm{f}\,\delta\mathrm{u}\,\mathrm{d}\Omega
\end{aligned} \tag{4.95}$$

Considering the vector identity

$$\overline{\nabla}\cdot\left(\alpha\,\overline{v}\right)=\overline{\nabla}\alpha\cdot\overline{v}+\alpha\overline{\nabla}\cdot\overline{v} \tag{4.96}$$

with $\alpha=\delta u$, $\overline{v}=\overline{\nabla}u$, and applying it to (4.95), one has:

$$\delta\chi=\int_{\Omega}p\,\overline{\nabla}\cdot\left(\delta u\,\overline{\nabla}u\right)d\Omega-\int_{\Omega}p\,\delta u\,\overline{\nabla}\cdot\overline{\nabla}u\,d\Omega-\int_{\Omega}f\,\delta u\,d\Omega \tag{4.97}$$

Using Gauss' theorem, (4.97) becomes:

$$\delta\chi=\int_{\Gamma}p\,\delta u\,\frac{\partial u}{\partial n}d\Gamma-\int_{\Omega}\delta u\left(p\overline{\nabla}\cdot\overline{\nabla}u+f\right)d\Omega \tag{4.98}$$

where n is the outward unit-vector normal to Γ.
Applying boundary conditions (4.92)-(4.93), one obtains:

$$\delta\chi=-\int_{\Omega}\delta u\left(p\overline{\nabla}\cdot\overline{\nabla}u+f\right)d\Omega \tag{4.99}$$

Since steadiness of energy is reached when $\delta\chi=0$ for any u, the following condition holds:

$$p\overline{\nabla}\cdot\overline{\nabla}u+f=0\ \text{ in }\Omega \tag{4.100}$$

Therefore, the minimisation of the field energy gives rise to the Poisson's equation with relevant boundary conditions; the latter is the Euler's equation of functional (4.94).
Conversely, starting form Poisson's equation, it is possible to obtain the variational equation (4.99) of energy functional (4.94). To this end, in the case of an elementary variation δu of potential u – variation which is supposed to fulfil boundary conditions –, the equation

$$\left(\overline{\nabla}\cdot p\overline{\nabla}u\right)\delta u+f\,\delta u=0 \tag{4.101}$$

is considered. Moving from the local viewpoint to the global one, one has:

$$\int_{\Omega}\left(\overline{\nabla}\cdot p\overline{\nabla}u\right)\delta u\,d\Omega+\int_{\Omega}f\,\delta u\,d\Omega=0 \tag{4.102}$$

Applying vector identity (4.96) to (4.102) with $\alpha=\delta u$, $\overline{v}=p\overline{\nabla}u$, it turns out to be:

$$\begin{aligned}&\int_{\Omega}\overline{\nabla}\cdot\left(\delta u\,p\overline{\nabla}u\right)d\Omega+\int_{\Omega}\overline{\nabla}(\delta u)\cdot p\overline{\nabla}u\,d\Omega+\int_{\Omega}f\,\delta u\,d\Omega=\\&=\int_{\Omega}\overline{\nabla}\cdot\left(\delta u\,p\overline{\nabla}u\right)d\Omega+\int_{\Omega}\delta\overline{\nabla}u\cdot p\overline{\nabla}u\,d\Omega+\int_{\Omega}f\,\delta u\,d\Omega=0\end{aligned} \tag{4.103}$$

Again, it is possible to apply Gauss' theorem to the first term of (4.103), so obtaining:

$$\int_{\Gamma} p\delta u \frac{\partial u}{\partial n} d\Gamma + \int_{\Omega} \delta \overline{\nabla} u \cdot p \overline{\nabla} u \, d\Omega + \int_{\Omega} f \, \delta u \, d\Omega = 0 \tag{4.104}$$

Due to boundary conditions, the firs term vanishes, so giving:

$$\int_{\Omega} \delta \overline{\nabla} u \cdot p \overline{\nabla} u \, d\Omega + \int_{\Omega} f \, \delta u \, d\Omega = 0 \tag{4.105}$$

or, equivalently:

$$\delta \left[\int_{\Omega} \frac{1}{2} p \overline{\nabla} u \cdot \overline{\nabla} u \, d\Omega \right] + \delta \left[\int_{\Omega} fu \, d\Omega \right] =$$

$$= \delta \left[\int_{\Omega} \left(\frac{1}{2} p \overline{\nabla} u \cdot \overline{\nabla} u - fu \right) d\Omega \right] = 0 \tag{4.106}$$

for any u. The term under integral is the total energy density of the field region (material + source), the variation of which is zero in steady conditions.

Overall, it has been proven that steadiness of energy is intrinsic to Poisson's problem and *vice versa*. In other words, solving a Poisson-like problem is equivalent to minimising the associated energy functional.

This result can be considered as a principle of unification between analysis and synthesis, because they give complementary tools to solve the same physical problem.

More generally, the following remarks can be put forward.

- It is possible to solve analysis problems by minimising a suitable energy functional; likewise, synthesis problems can be solved through the minimisation of an objective function expressing a performance criterion. So, at least from the theoretical viewpoint, minimisation is a general tool for problem solving.
- In a sense, all field-dependent inverse-problems might be considered multiobjective in their essence, because it is always possible to solve an analysis problem by minimising the associated energy-functional and, similarly, a synthesis problem can be reduced to the minimisation of one or more objective functions.