An Introduction to Inverse Problems and Optimisation in Electromagnetism

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Reference textbooks

- P. Di Barba, A. Savini, S. Wiak: Field Models in Electricity and Magnetism. Springer, 2008
- P. Neittaanmäki, M. Rudnicki, A. Savini: Inverse Problems and Optimal Design in Electricity and Magnetism. Clarendon Press, 1996
- P. Di Barba: Multiobjective Shape Design in Electricity and Magnetism. Springer, 2010

A Copernican revolution: from direct to inverse problems

- In engineering science, direct problems are defined as those where, given the input or the cause of a phenomenon or of a process in a device, the purpose is that of finding the output or the effect.
- Conversely, inverse problems, are those where, given the measured or expected output or effect, one wants to determine the input or the cause.
- The two types of problems, when applied to the same phenomenon or process, represent the two logical ways of conceiving it: from input to output or the other way round.

Classification of inverse problems

In electromagnetism, inverse problems may appear in either of two forms:

- given measured data in a field region, to recover the relevant sources or boundary conditions or material properties (identification or parameter-estimation problems);
- given the prescribed field in a device, to determine sources or b.c. or materials or shape of the device, producing the specified performance (synthesis or optimal design problems).

Insidiousness of inverse problems

From the mathematical viewpoint, following the Hadamard definition (1923), well-posed problems (or properly, correctly posed problems) are those for which:

- a solution always exists;
- there is only one solution;
- a small change of data leads to a small change in the solution.

Ill-posed problems, instead, are those for which:

- a solution may not exist;
- there may be more than one solution;
- a small change of data may lead to a big change in the solution.

The last property implies that the solution does not depend continuously upon the data, which often are measured quantities and therefore are affected by noise or error.

Ill-posed problems: remarks

- Identification problems have always a solution at least, while a solution may not exist for optimal design problems; this happens when *e.g.* the prescribed quantity does not fit with the data.

- On the contrary, if multiple solutions exist to a given problem, they might be similar, differing by e.g. a degree of agreement of field model to supplied data.

- Even if the agreement is very good, it might happen that the solution is unstable: a small perturbation in the data causes a large oscillation in the solution.

All these reasons make inverse problems more insidious than direct problems.

Inverse problems and design problems

- Any design problem can be formulated in mathematical terms as an inverse problem.
- In particular, optimal shape design problems, which are very popular in all branches of engineering, belong to a group of inverse problems where the purpose is to find the geometry of a device which can provide a prescribed behaviour or an optimal performance.
- The ultimate goal is to perform an automated optimal design (AOD), when the solution is obtained automatically in terms of the required or best performance.

Solving an inverse problem by minimising a functional

- In general, the n_v unknowns x of an inverse problem are called design variables or degrees of freedom. The design variables may be geometric coordinates of the field region or values of sources or parameters characterizing the region.

- The solution to an inverse problem is generally performed by means of the minimisation of a suitable function f(x) called objective function, or cost function, or design criterion. This function may represent some performance depending on the field, or simply the residual between computed and prescribed field values (error functional).

- In mathematical terms, the problem reads:

  given $x_0 \in \Omega \subseteq R^{n_v}$

  find $\inf_x f(x) \quad , \quad x \in \Omega \subseteq R^{n_v}$

  where x_0 is an initial guess. Properly speaking, it is a problem of unconstrained minimisation; to be more meaningful, it is assumed that f is limited in Ω.

Constrained minimisation

- The objective function should fulfil constraints, which may be expressed as equalities, inequalities and side bounds. Formally, the problem can be stated as follows:

given $\quad x_0 \in \Omega \subseteq R^{n_v}$

find $\quad \inf_x f(x)\ ,\ x \in \Omega \subseteq R^{n_v}$

subject to

$$g_i(x) = 0 \quad , \quad i = 1,\dots,n_c$$

$$h_j(x) \le 0 \quad , \quad j = 1,\dots,n_e$$

$$\ell_k \le x_k \le u_k \quad , \quad k = 1,\dots,n_v$$

- Constraints and bounds set the boundary of the feasible region Ω associated with function f(x), and define implicitly its shape in the n_v-dimensional design space.

Insidiousness of minimisation

- Classical optimality requires the following first-order necessary condition, better known as Kuhn-Tucker theorem (1951):

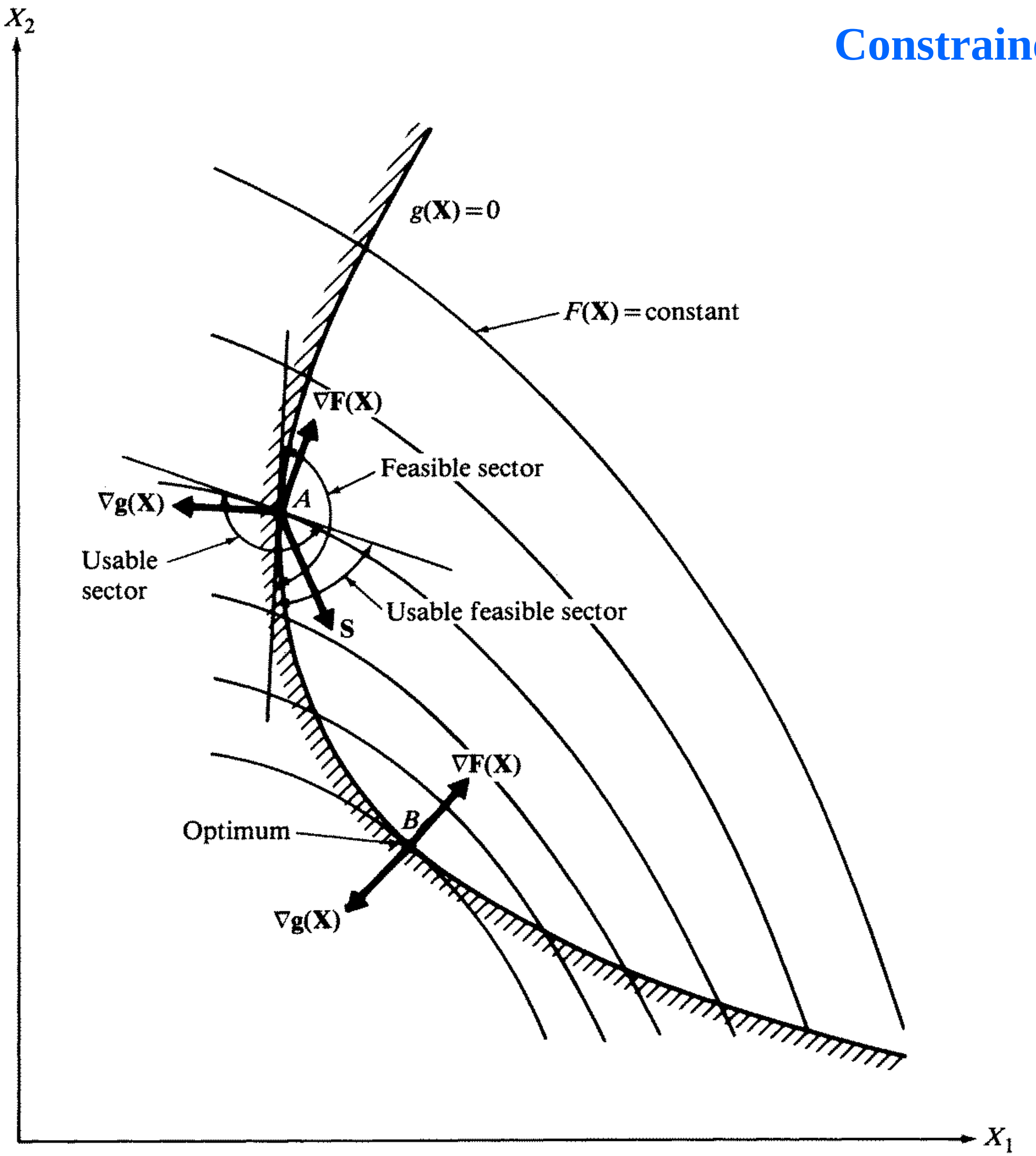
  Let $\tilde{x}$ be a local minimum point for f(x) and let f, g_i, h_j differentiable functions. Then, there exists a vector $\tilde{\lambda} \in \Re^{n_c + n_e}$ of multipliers such that

$$\overline{\nabla} f(\tilde{x}) + \sum_{i=1}^{n_c} \tilde{\lambda}_i \overline{\nabla} g_i(\tilde{x}) + \sum_{j=1}^{n_e} \tilde{\lambda}_{j+n_c} \overline{\nabla} h_j(\tilde{x}) = 0$$

$$\tilde{\lambda}_i g_i(\tilde{x}) = 0 \ , \ \tilde{\lambda}_i \geq 0$$

- This is a sufficient condition for $\tilde{x}$ to be a global minimum point if f(x) is a convex function and Ω is a convex region.

Constrained minimization



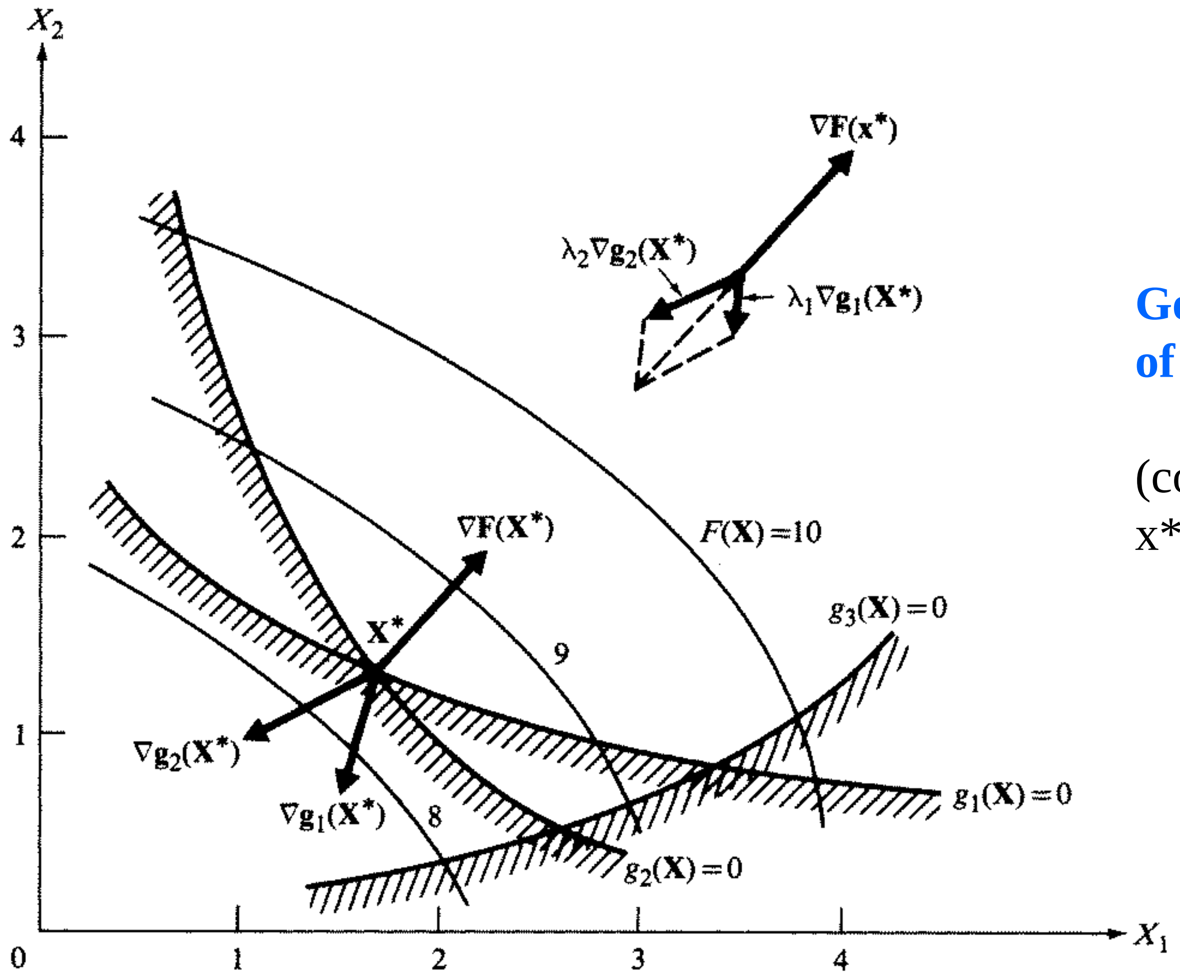
Usable and feasible search direction **S**

$$\bar{\nabla} f(A)\cdot \bar{S} \le 0$$

$$\bar{\nabla} g(A)\cdot \bar{S} \le 0$$

$$\bar{\nabla} f(B)\cdot \bar{S} = 0$$

$$\bar{\nabla} g(B)\cdot \bar{S} = 0$$



Geometric interpretation of KT conditions

(constraint g_3 is not active in x* and therefore λ_3=0)

Insidiousness of minimisation (2)

- In computational electromagnetism, it happens that functions f, g_i and h_j are known only numerically as a set of values at sample points; therefore, classical assumptions about differentiability and convexity cannot be assessed.

- In particular, when the assumption of convexity is not applicable, f might exhibit some local minima in addition to the global minimum.

- Moreover, the numerical approximation of the gradient is time consuming; moreover, it is a potential source of fatal inaccuracies.

An alternative: evolutionary computing

- Darwinian evolution is intrinsically a robust search; it has become the model of a class of optimisation methods for the solution of real-life problems in engineering.

- The natural law of survival of the fittest in a given environment is the model to find the best design configuration fulfilling given constraints.

- The principle of natural evolution inspired a large family of algorithms that, through a procedure of self-adaptation in an intelligent way, lead to an optimal result (Goldberg, 1989).

Evolutionary computing (2)

- A primary advantage of evolutionary computing is its conceptual simplicity.
- A very basic pseudo-code of a typical algorithm:

i) initialize a population of individuals;
ii) randomly vary individuals;
iii) evaluate fitness of each individual;
iv) apply selection;
v) if the terminating criterion is fulfilled
then stop, else go to step ii).

Evolutionary computing (3)

Evolutionary computing algorithms are:

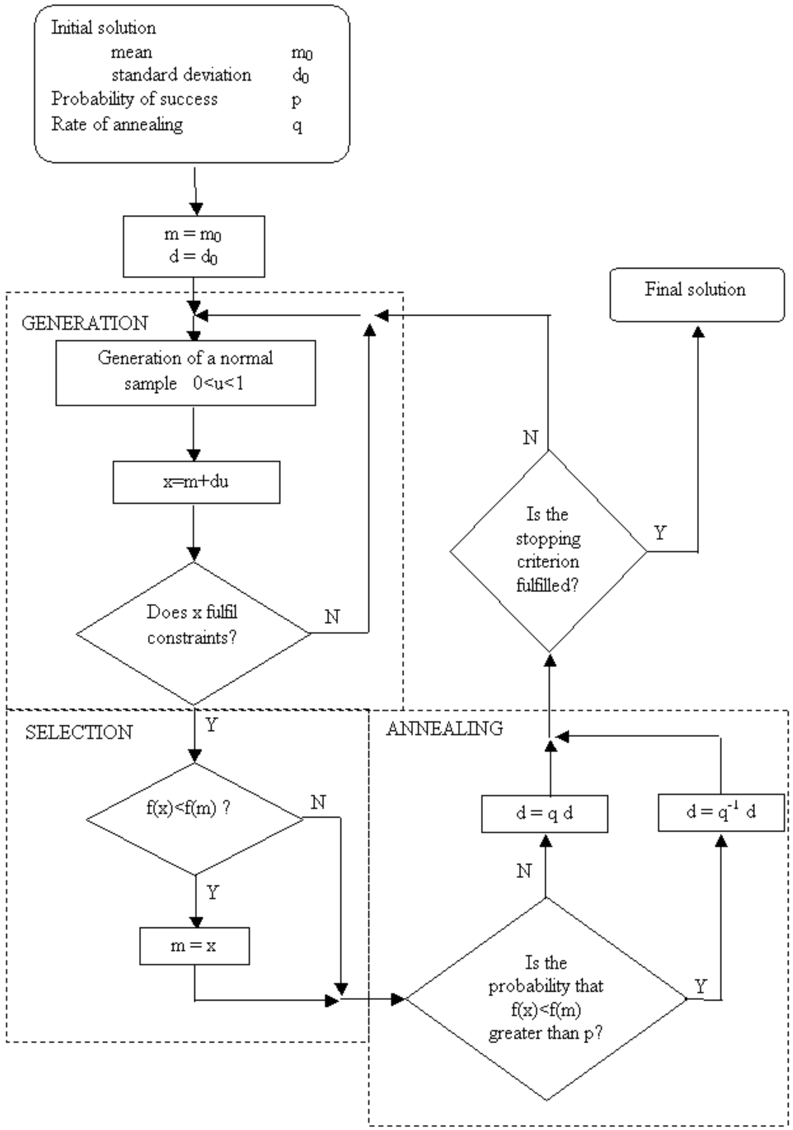
- easy to implement;
- gradient-free;
- global-optimum oriented.

On the other hand, they are rather slow and costly, because of the high number of function evaluations required to converge.

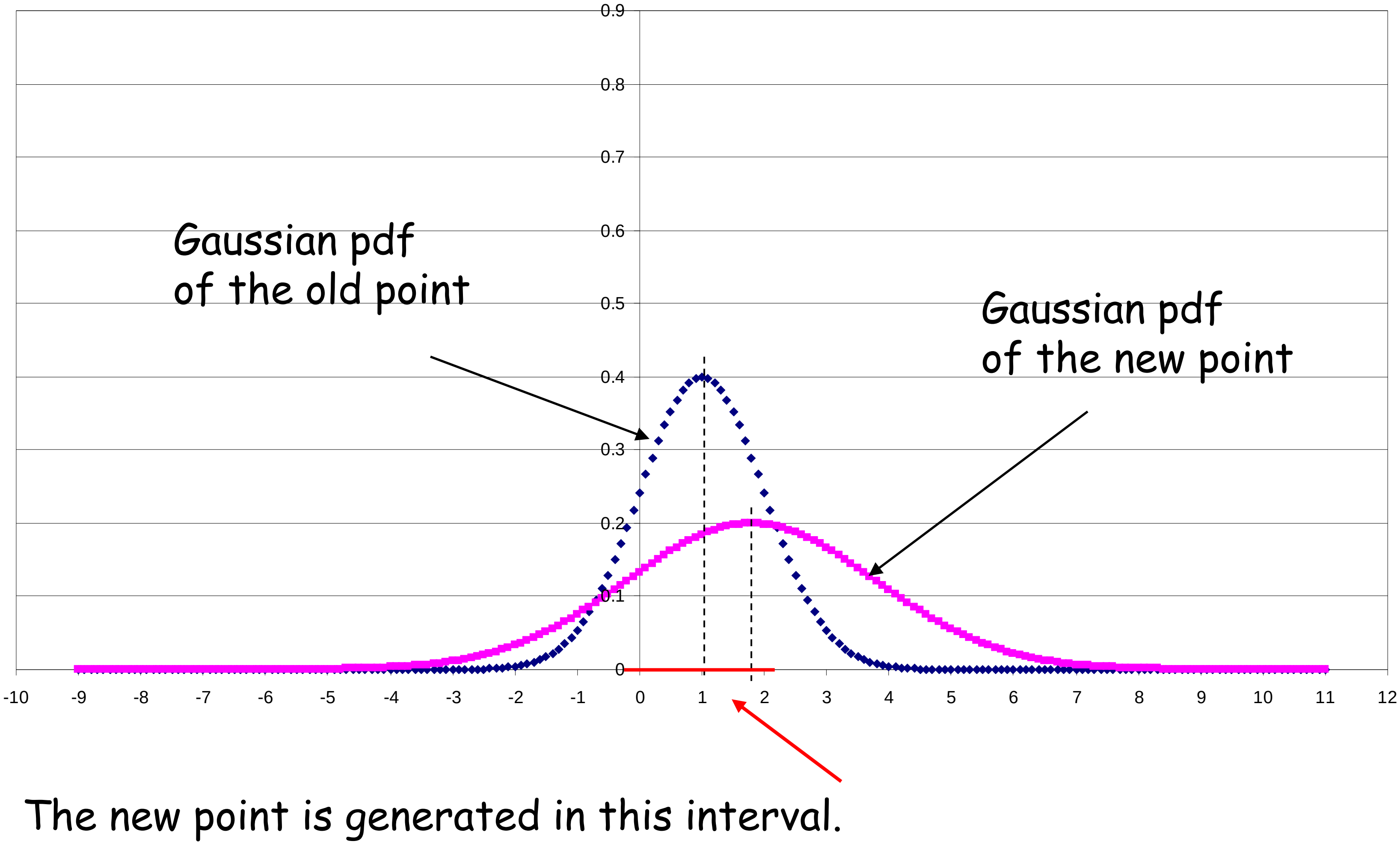
An evolution strategy of lowest order

Evolution strategy mimics the survival of the fittest individual that is observed in nature.

The flow-chart of an algorithm (in which a single parent generates a single offspring) is here presented.

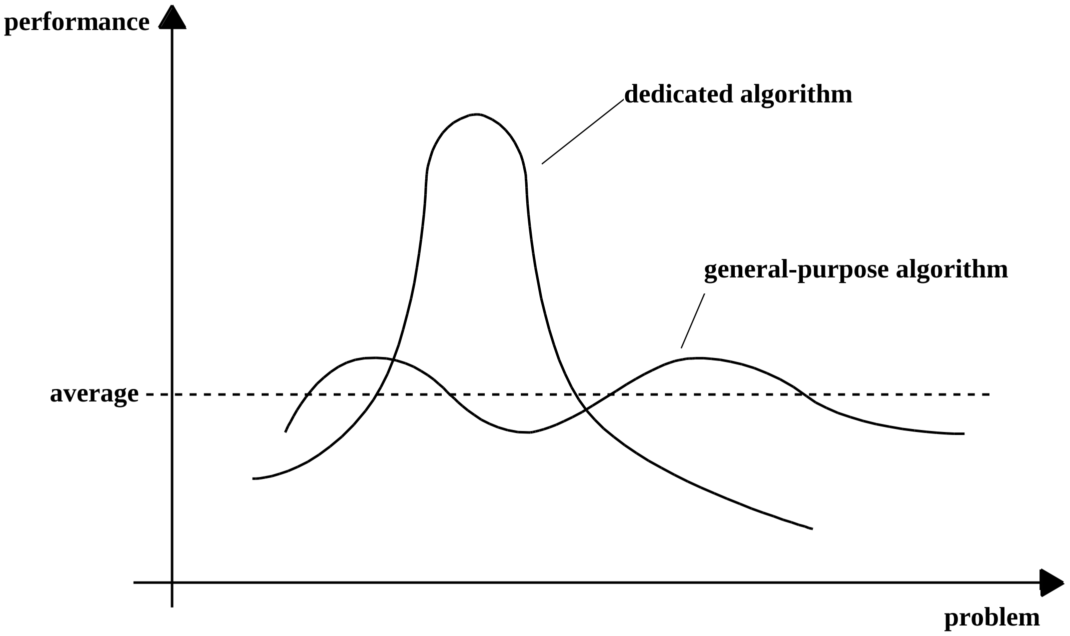


Generation of a new design point



Insidiousness of minimisation (3)

The No-Free Lunch (NFL) theorem (Wolpert and Macready, 1997)



There is no best algorithm, whether or not it is evolutionary

Whatever an algorithm gains in performance on a class of problem, is necessarily lost by the same algorithm in the other problems

FIELD-BASED OPTIMAL SHAPE DESIGN

Design vector x represents the geometry of the device to be synthesized.

Generally, j-th objective function f_j , j = 1,n_f is a field-dependent quantity.

The following mapping applies: $\text{geometry } \{x\} \rightarrow \text{field } s(x) \rightarrow \text{objective } f_j(x, s(x)) \ , \ j = 1, n_f$

finite element analysis ← Maxwell equations

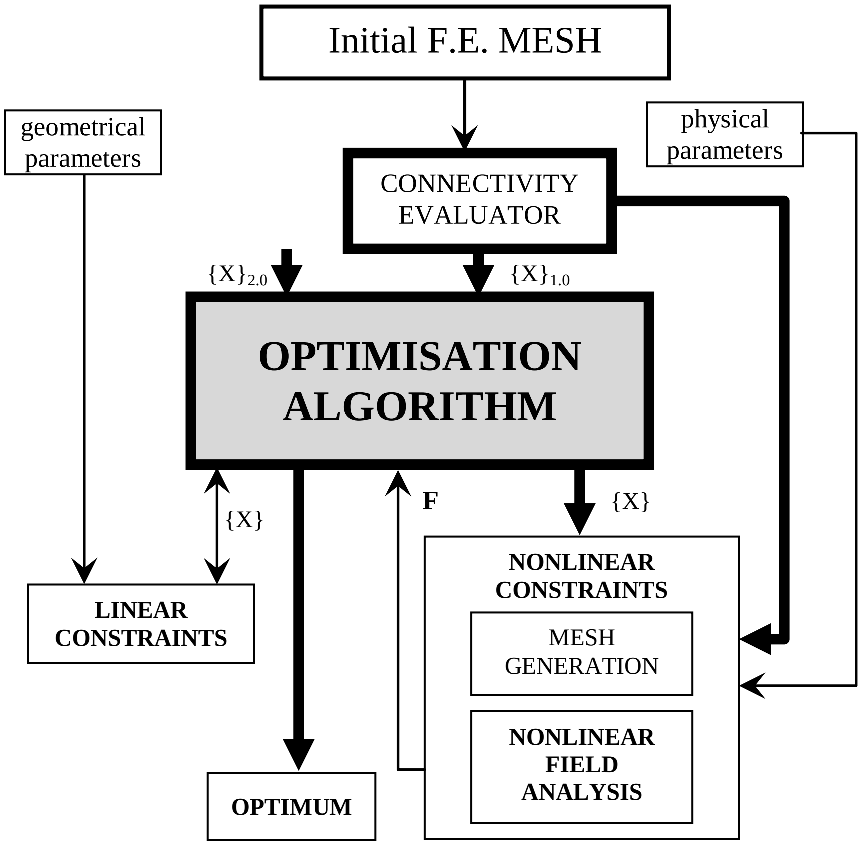
The minimisation problem reads: find $\inf_x \ f_j(x, s(x)) \ , \ x \in \Omega \subset \Re^{n_v} \ , \ j = 1, n_f$

subject to n_c field-dependent constraints $C = \{x | g_k(x, s(x)) \leq c_k \in \Re \ , \ k = 1, n_c\}$

In a problem of shape design, two aspects are always involved: the optimal synthesis of field s which takes place in the device, and the optimal design of device geometry x.

Numerical solution to design problems: AOD

A procedure of AOD requires, as a rule, a routine for calculating the field, which is integrated in a loop with a routine optimising the objective function.



Numerical solution to design problems (2)

- The device to be optimised is represented by a numerical model in two or three dimensions (i.e. a grid of nodes and elements).
- The main flow of the computation is driven by the optimisation routine (e.g. evolution strategy). Starting from x_0 , an iterative procedure updates the current design point x_k in x_{k+1} .
- Given x_{k+1} , the routine of field analysis generates a new finite-element grid, the field simulation is restarted and the evaluation of f(x) is so updated.
- If the procedure converges, the result could represent either a local minimum or the global minimum or simply a point better than the initial one, because f has decreased; in the latter case, a mere improvement (and not the optimisation) of f has been achieved.

Numerical solution to design problems (3)

- Usually, the analysis of field can be performed either by differential methods originated from Maxwell equations (finite-difference method, finite-element method), or by integral methods derived from Green theorem (boundary-element method).

- In turn, numerical optimisation can be achieved by means of deterministic (i.e. gradient-based) methods or evolutionary (i.e. gradient-free) methods.

- Nowadays, most of commercially available codes devoted to electromagnetic field simulation are based on the finite-element analysis (FEA): they proved, in fact, to offer a general-purpose and flexible tool of field simulation.

Numerical solution to design problems (4)

- Commercial FEA codes are equipped with a user interface, which enables the designer to develop a model in two or three dimensions by means of graphical operations only.
- These features make the simulation environment rather friendly and easy to use; so, in practice, FEA has become the most popular tool, mainly in an industrial centre for R&D.
- The combination of any method for analysis and any method for minimisation gives origin to a variety of iterative procedures for solving an optimal design problem.

Multiobjective formulation of a design problem

Often, in electromagnetic design, multiple objective functions should be optimised simultaneously.

These problems belong to the category of multi-objective or multi-criteria. Their formulation is characterized by a vector of objective functions.

Formally, considering n_v variables and n_f objectives, one has:

given $x_0 \in \Re^{n_v}$, find $\inf_x F(x)$, $x \in \Re^{n_v}$, $F \in \Re^{n_f}$

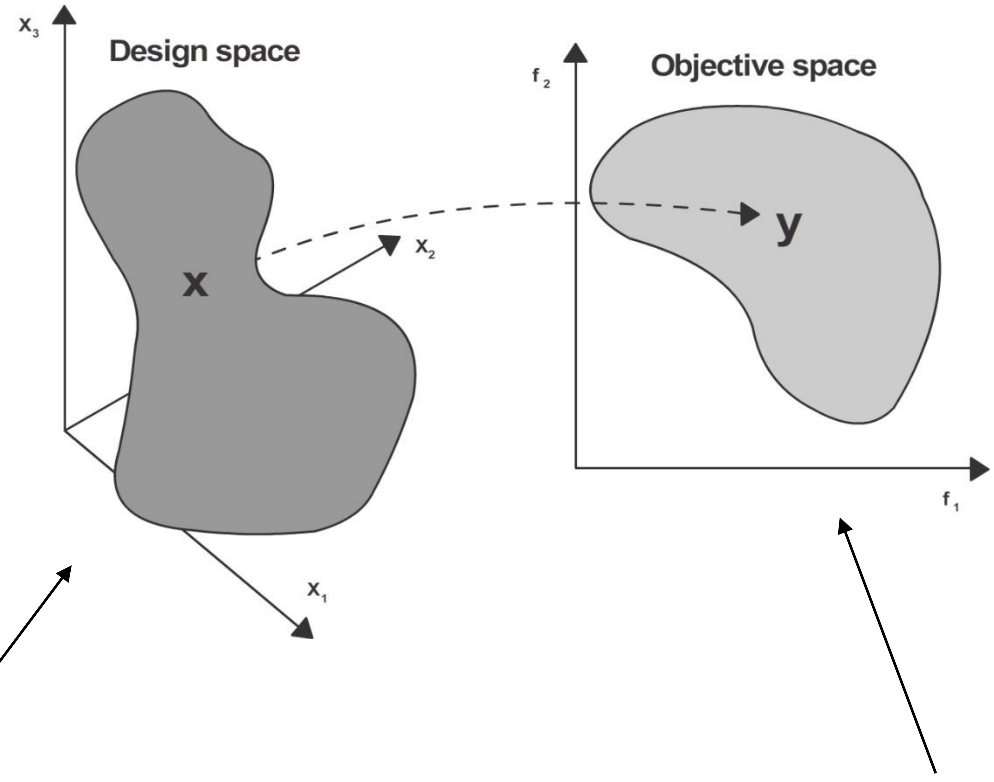
subject to n_c inequality and n_e equality constraints

$$g_i(x) \le 0 \; , \; i = 1, n_c \qquad h_j(x) = 0 \; , \; j = 1, n_e$$

and to $2n_v$ side bounds

$$\ell_k \le x_k \le u_k \; , \; k = 1, n_v$$

Mapping from design space to objective space



n_v - dimensional

n_f - dimensional

Preference function formulation

Traditionally, the multiobjective problem is reduced to a single-objective one by means of a preference function $\psi(x)$, *e.g.* the weighted sum of the objectives:

$$\psi(x) = \sum_{i=1}^{n_f} c_i f_i(x)$$

with $0 < c_i < 1$, $\sum_{i=1}^{n_f} c_i = 1$

$\psi(x)$ should be minimised with respect to $x \in \Re^{n_v}$
subject to the problem contraints.

The hierarchy attributed to the i-th objective can be modified by changing the corresponding weight c_i . For a given set of weights, the relevant solution, if any, is assumed to be the optimum.

Paretian optimality (1)

The most general solution to the design problem is given by the front of

Pareto-optimal solutions

⇕

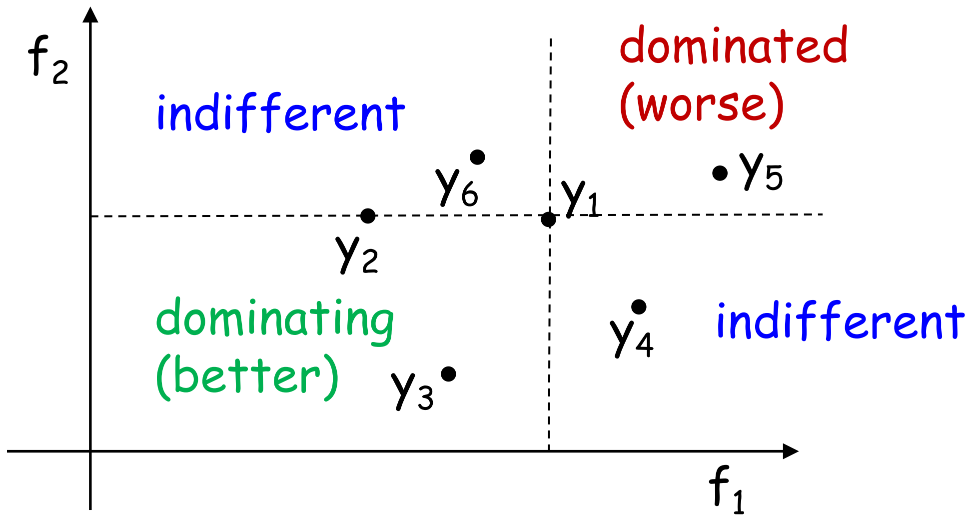
Solutions for which the decrease of an objective is not possible without the simultaneous increase of at least one of the other objectives

⇓

This means to have a family of solutions to be compared

Paretian optimality (2)

A solution is said to dominate another one if the first is better than the second with respect to one objective, without worsening all the other objectives.



Two solutions are indifferent to each other if the first is better than the second for some objectives, while the second is better than the first in all the other objectives.

Paretian optimality (3)

Given two solutions x_j and x_k

situation	consequence
x_j dominates x_k	x_j is better than x_k
x_k dominates x_j	x_k is better than x_j
none of the two	x_j and x_k are indifferent

Two key definitions

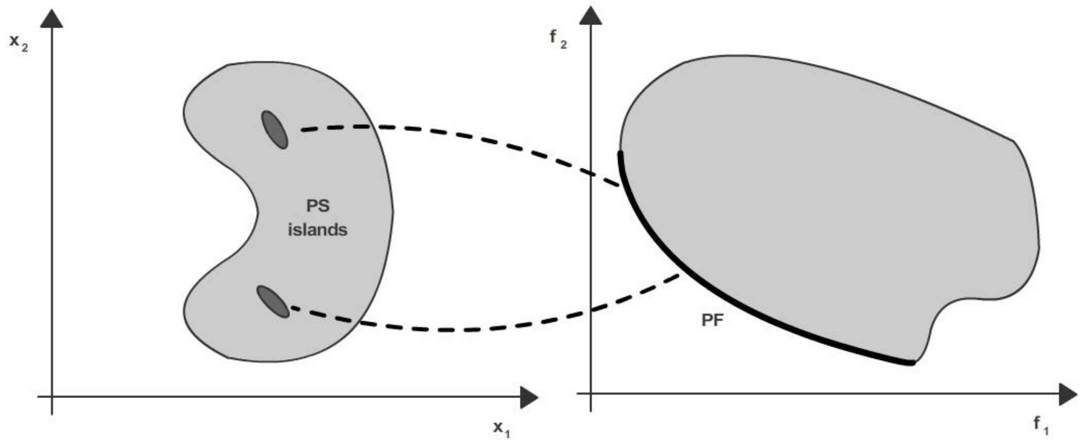
Let $Y \subseteq \Re^{n_f}$ be an objective space. Then, a point $y \in Y$ is said to be Pareto optimal if no point $\tilde{y} \in Y$ exists such that $F^{-1}(\tilde{y})$ dominates $F^{-1}(y)$.

Let $F(x): X \rightarrow Y$ be a vector of n_f objectives, with design space $X \subseteq \Re^{n_v}$ and objective space $Y \subseteq \Re^{n_f}$: the set $\Phi = \{ y \in Y \mid y \text{ is Pareto optimal} \}$ is the Pareto front (PF); the set $\Xi = \{ x \in X \mid F(x) \in \Phi \}$ is the Pareto set (PS).

Correspondence between PF and PS

In practice, the objective space Y is the control space

Metric criteria to identify non-dominated solutions in the design space X



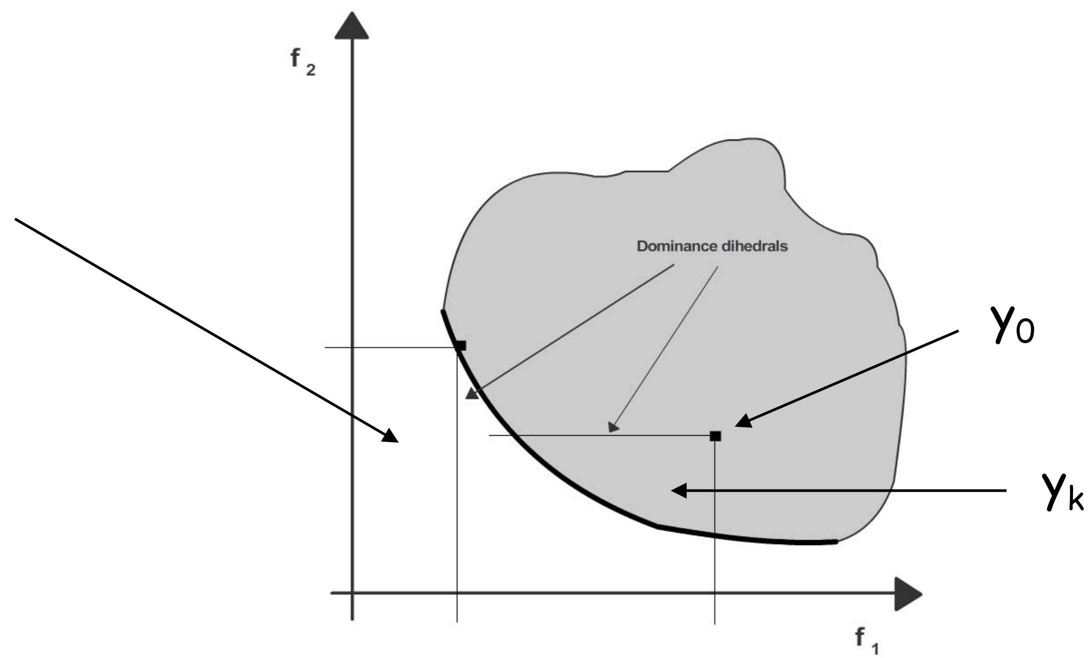
The PS topology depends on the Y to X inverse mapping:
it might form e.g. a set of islands

Dominance dihedral

Orthogonal sector in the objective space:

- has its vertex at a given point y_0 ;
- contains all the points y_k such that $F^{-1}(y_k)$ dominates $F^{-1}(y_0)$.

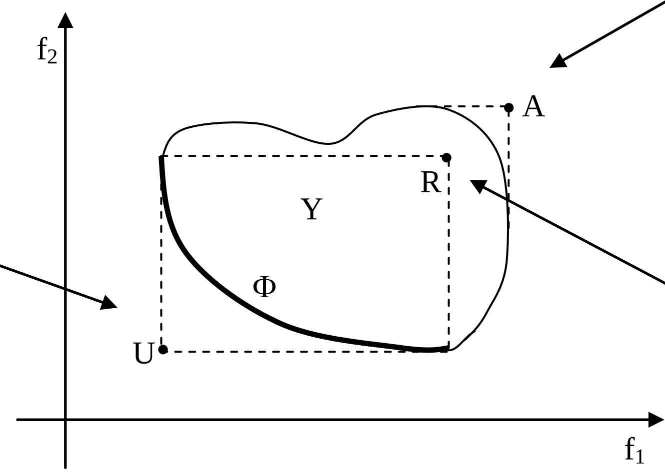
If the dihedral is empty, then $F^{-1}(y_0)$ is said to be non-dominated.



The objective space: a geometric interpretation in 2D

anti-utopia point

utopia point



nadir point

ideal objective vector

$$U = \left(U_1, \ldots, U_i, \ldots, U_{n_f}\right)$$

$$U_i = \inf_x f_i(x)\ ,\ i = 1, n_f$$

If $U_i = U_{i+1}$, $i=1, n_f - 1$ then the optimisation problem is single-objective

$$f_i(\tilde{x}_i) = U_i\ ,\ \tilde{x}_j \neq \tilde{x}_{j+1}\ ,\ i = 1, n_f\ ,\ j = 1, n_f - 1$$

A conflict exists

HIGHER-ORDER DIMENSIONALITY $n_f>2$

Metric matrix ($n_f=3$)

$$M = \begin{bmatrix} U_1 & f_2\big|_{f_1=U_1} & f_3\big|_{f_1=U_1} \\ f_1\big|_{f_2=U_2} & U_2 & f_3\big|_{f_2=U_2} \\ f_1\big|_{f_3=U_3} & f_2\big|_{f_3=U_3} & U_3 \end{bmatrix}$$

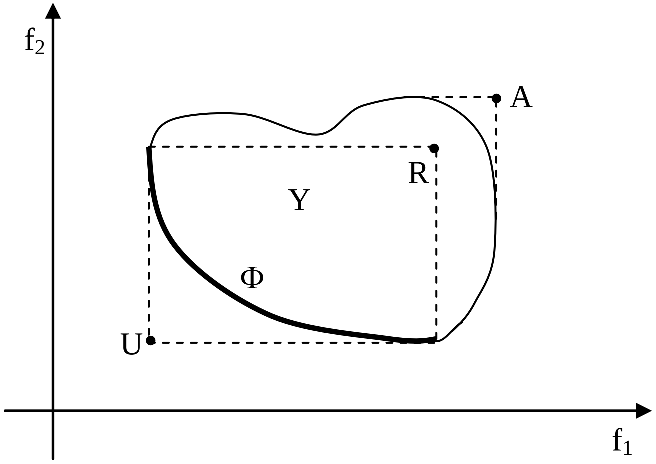
n_f SO optimisations

Utopia point

Nadir point

$$R_i = \max_{j=1,n_f} M_{ij} \;, \; i = 1, n_f$$

PREFERENCE FUNCTIONS



Scalarizing distance $\left\| R - U \right\|$

Possible preference functions

$$\psi(x) = \sum_{i=1}^{n_f} \frac{w_i f_i(x)}{R_i - U_i} \ , \ \sum_{i=1}^{n_f} w_i = 1 \ , \ w_i \in \Re^+$$

$$\psi(x) = \sum_{i=1}^{n_f} \frac{w_i f_i(x) - g_i(x)}{R_i - U_i} \ , \ \sum_{i=1}^{n_f} w_i = 1 \ , \ w_i \in \Re^+$$

$g_i(x)$ user-defined goal

n_f+1 SO optimisations → 1 Pareto-optimal solution

Open problem:

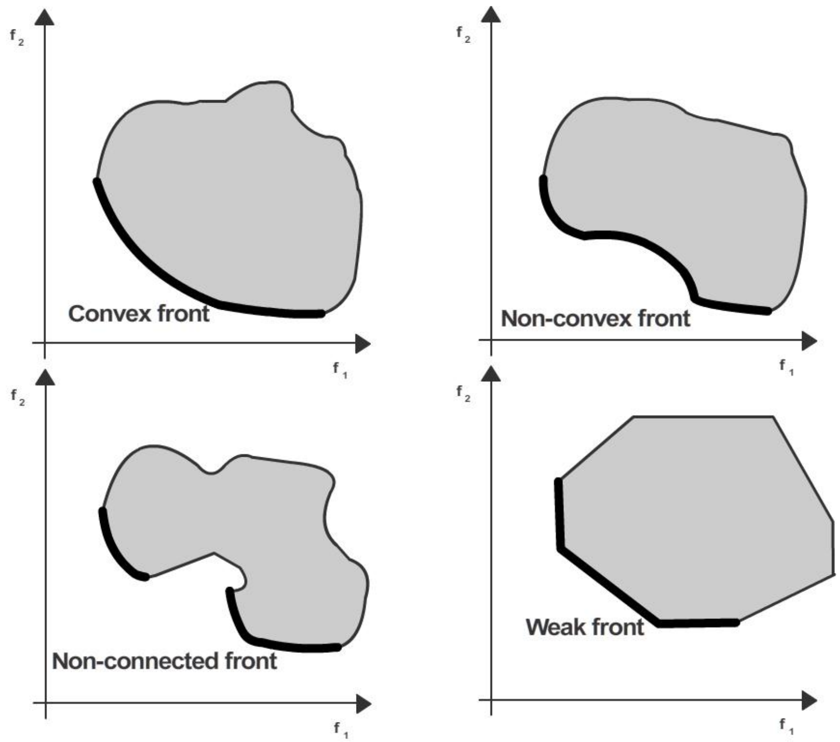
spacing of Pareto-optimal solutions along the front

Geometric classification of the PF in 2D (1)

PF as a function of
of n_f -1 objectives:

$$\tilde{f}_{n_f} = \tilde{f}_{n_f}\left(f_1, \ldots, f_{n_f - 1}\right)$$

Typical topologies
of front

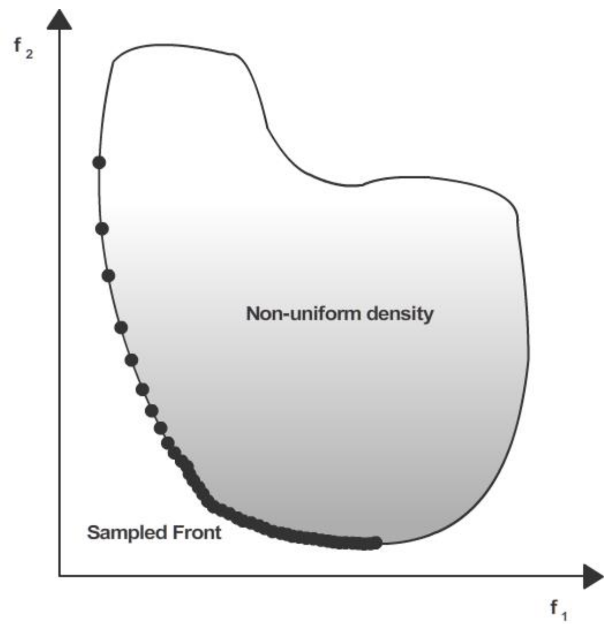


Ad hoc optimisation algorithms

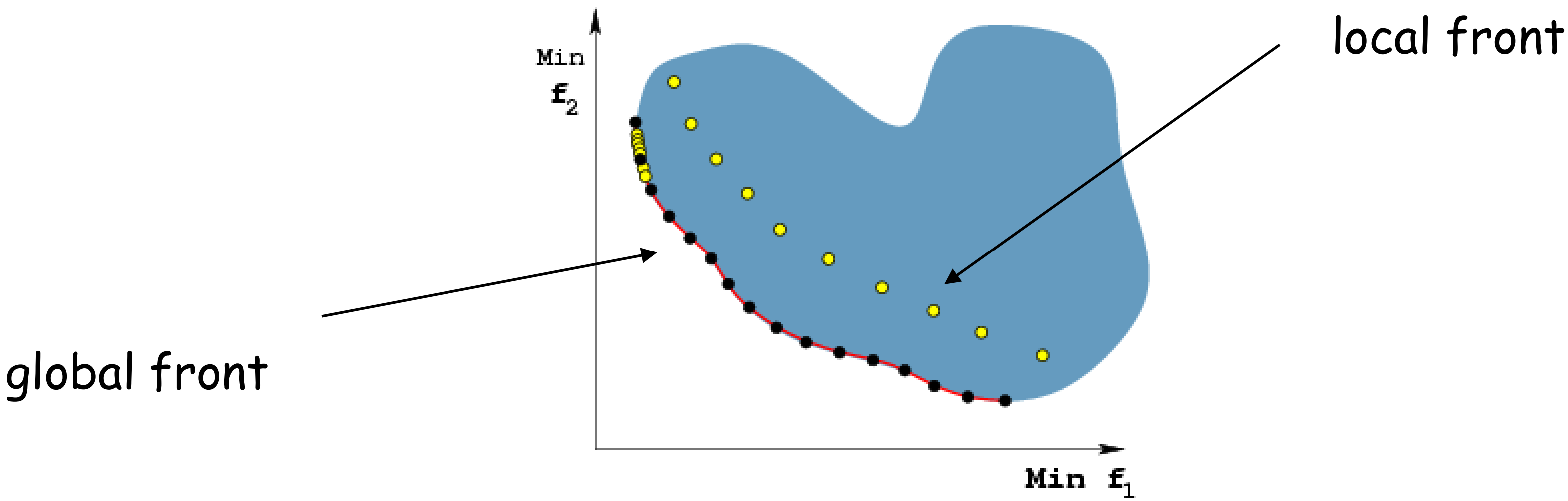
Geometric classification of the PF in 2D (2)

Non-uniformly sampled PF
(deceptive topology)

Non-linear objective functions



Multimodal optimisation problems



non-convex objective functions

local fronts, in addition to the global front

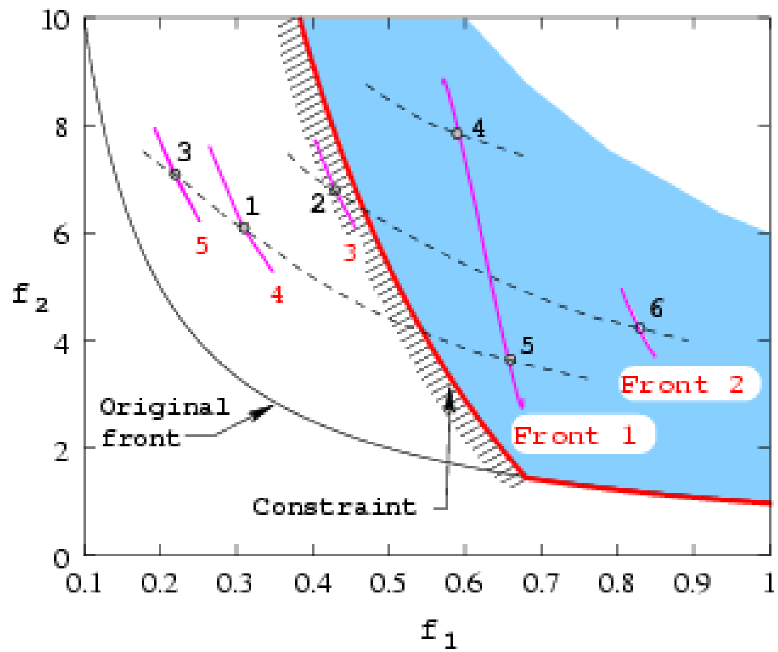
Handling constraints

Penalty function method (conventional) $\varphi(x) = \psi(x) + \sum_{i=1}^{n_c} r_i [g_i(x)]^2 \ , \ r_i > 0$

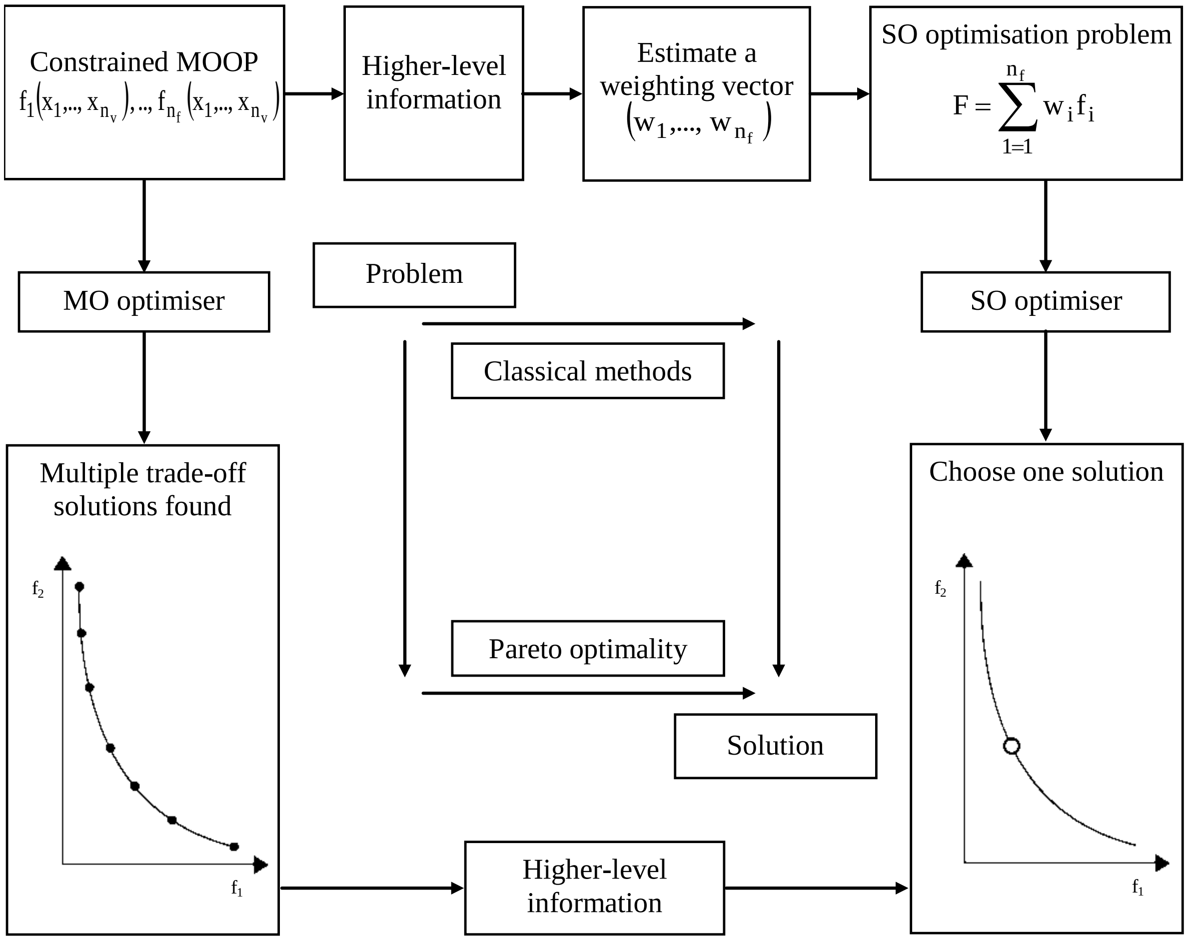
Constraint-dominance principle (parameter-less)

A solution *j* constraint-dominates a solution *k*, if any is true:

- *j* is feasible and k is not;
- *j* and *k* are both infeasible, but *j* has a smaller constraint violation;
- *j* and *k* are feasible and *j* dominates *k*.



Logical paths of classical and Paretian formulations

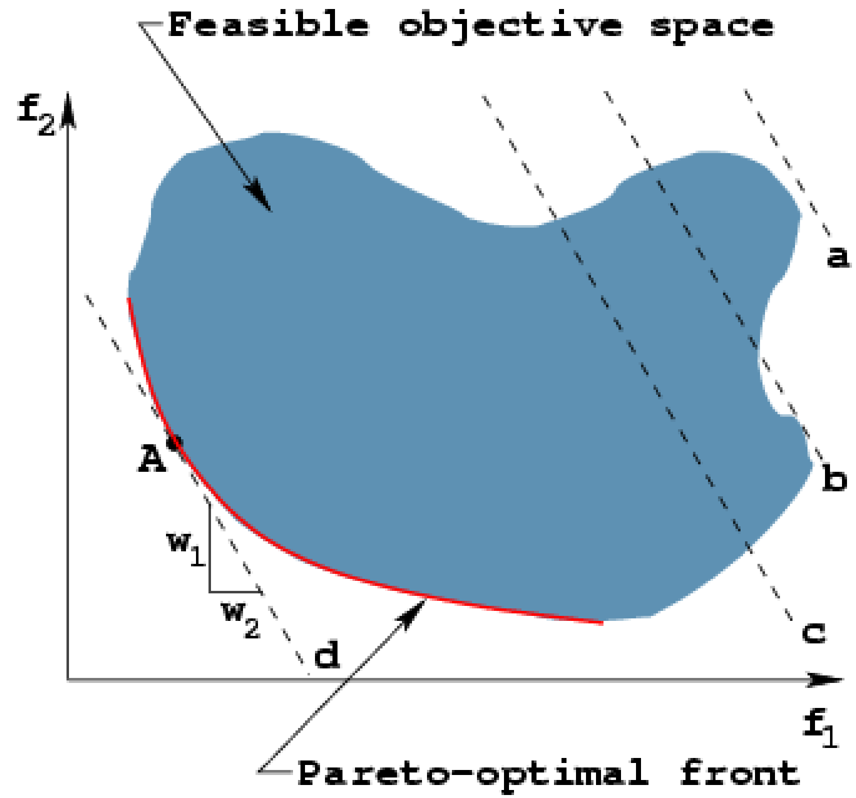


Classical Approach: Weighted Sum Method

- Construct a weighted sum of objectives and optimize

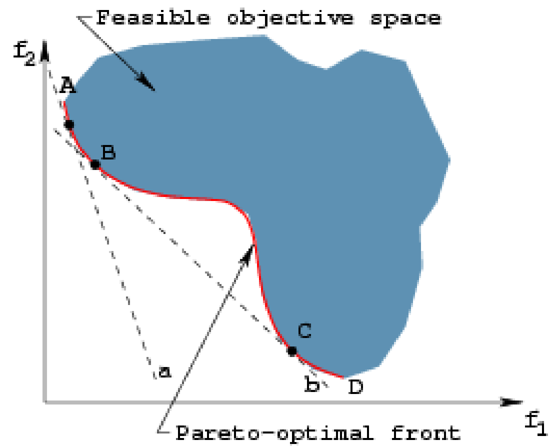
$$F(x) = \sum_{i=1}^{M} w_i f_i(x)$$

- User supplies weight vector *w*



Difficulties with Classical Methods

- Nonuniformity in Pareto-optimal solutions
- Inability to find some solutions
- Epsilon-constraint method still requires an $\mathcal{E}$ -vector

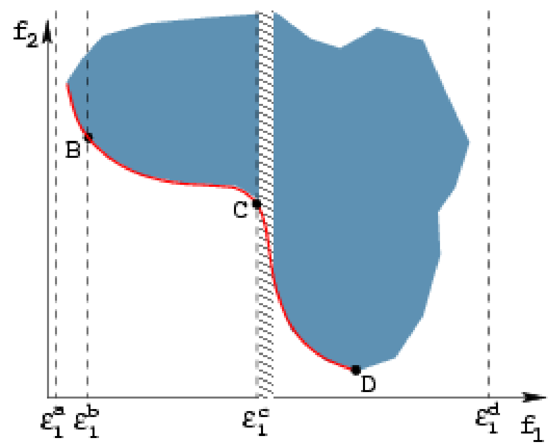


The ε-formulation reads:

given a set of n_f-1 values $\{\varepsilon_j\}$, $\varepsilon_j \in \Re$

find $\inf_x f_i(x)$, $x \in \Omega \subseteq R^{n_v}$

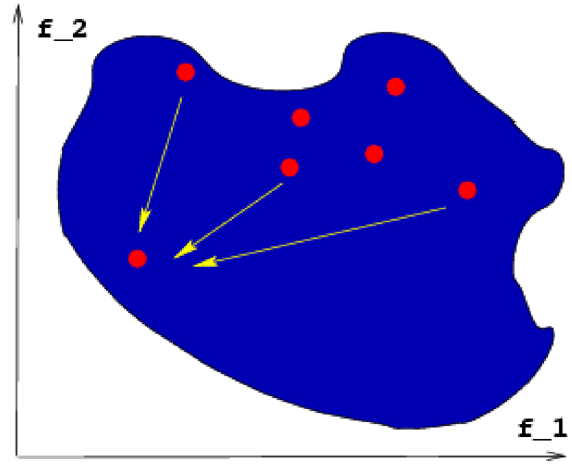
subject to $f_j(x) \leq \varepsilon_j$, $j \neq i$, $j = 1, n_f$

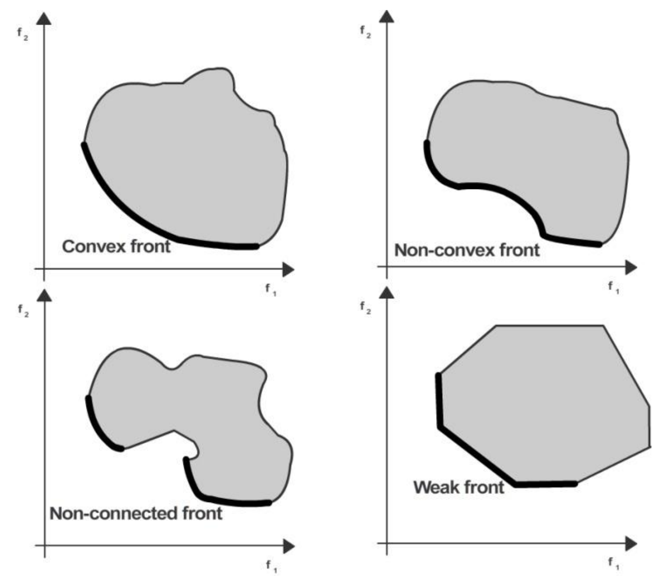


USING EVOLUTIONARY ALGORITHMS

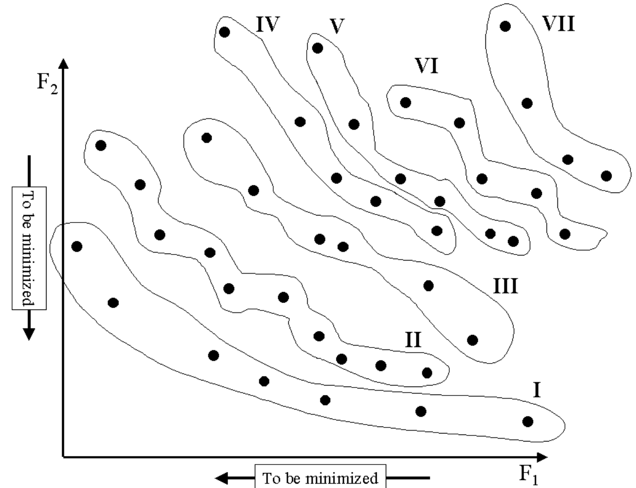
- Population approach suits well to find multiple solutions
- Niche-preservation methods can be exploited to find diverse solutions
- Implicit parallelism helps provide a parallel search

- Shape of Pareto front is not a matter (e.g. non-convexity, disconnectedness)

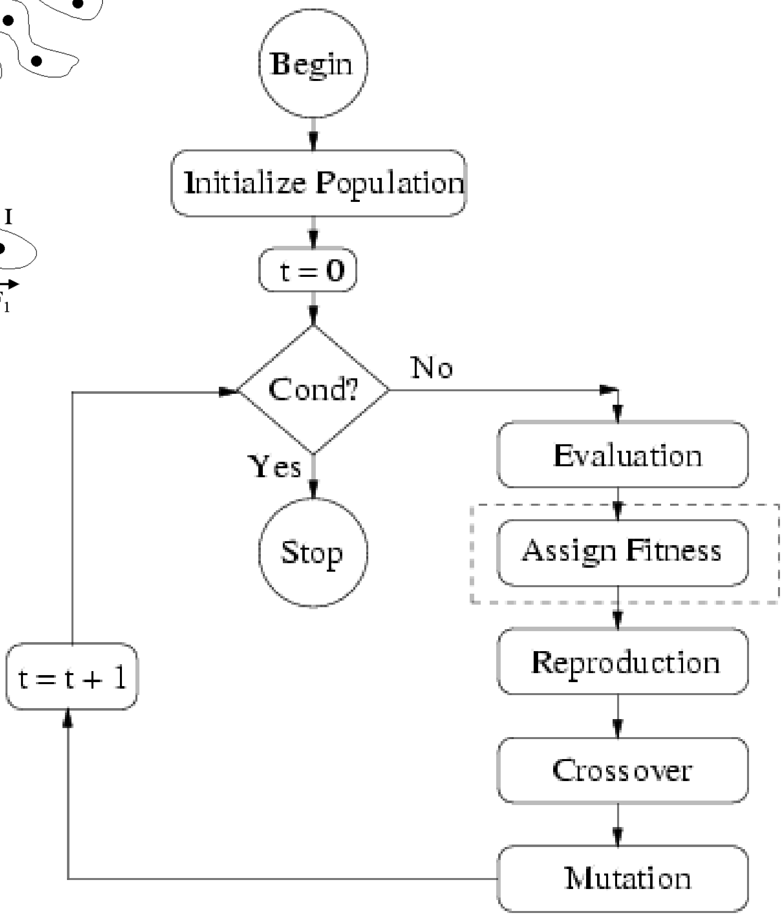




POPULATION-BASED APPROACH
WHAT TO CHANGE IN A BASIC GA ?



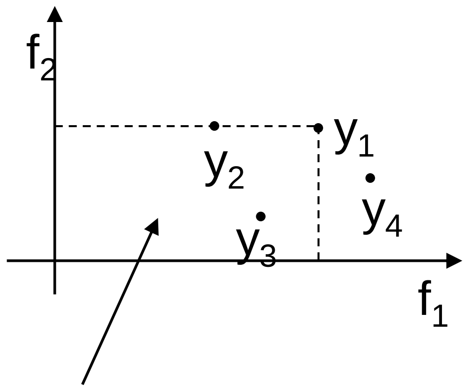
- Modify the fitness computation
- Emphasize non-dominated solutions for convergence
- Emphasize less-crowded solutions for diversity



Elitist Non-dominated Sorting Genetic Algorithm (NSGA-II)

INDIVIDUAL-BASED APPROACH
WHAT TO CHANGE IN A BASIC ESTRA ?

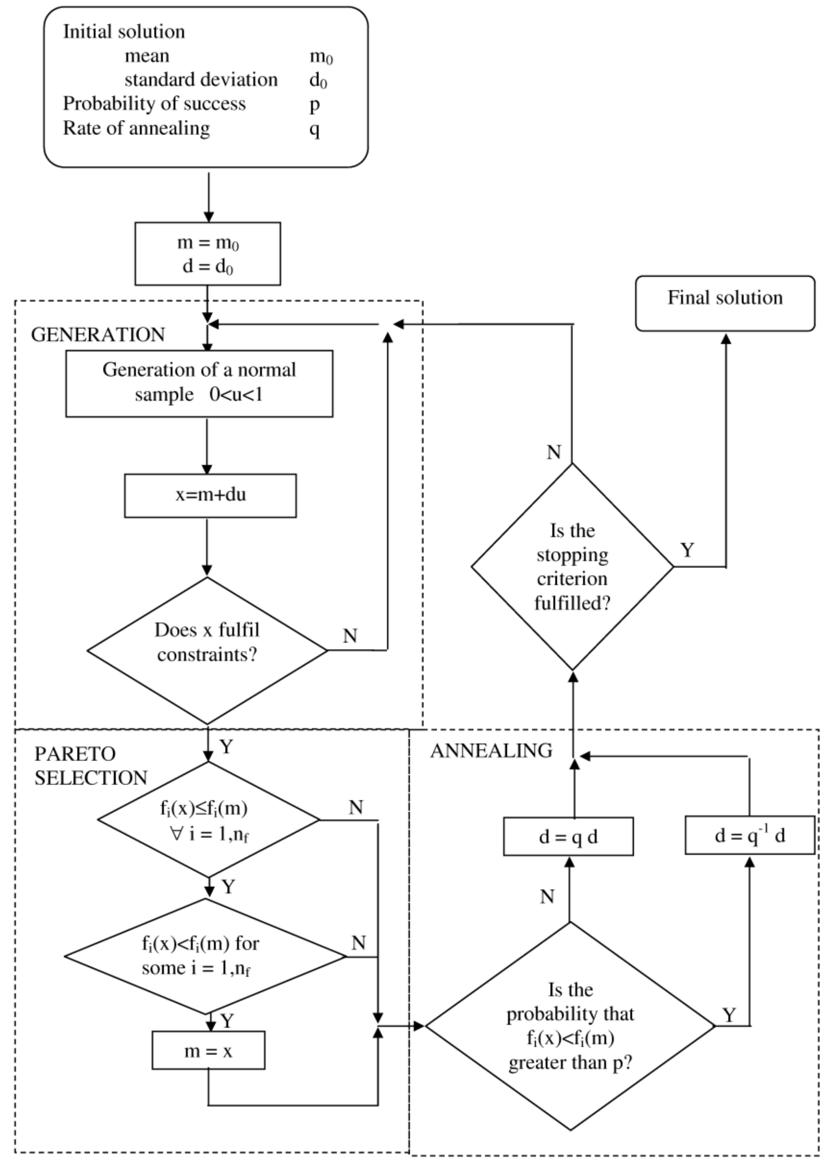
Modify the acceptance criterion of the offspring



domination dihedral wrt y_1

(y_2 , y_3) are accepted, y_4 is rejected

Multiobjective Evolution Strategy (MOESTRA)



PRACTICAL METHODS TO SOLVE EMO IN ELECTROMAGNETISM

Two main streams can be observed

- use approximation techniques
- identify a surrogate model of objectives and constraints

then

- use an evolutionary algorithm to optimize

- preserve the use of FEA (very flexible !) to solve the direct problem, but
- reduce the solution time of field analysis
- implement cost-effective strategies

⇩

well suited for an industrial R&D centre

SURROGATE MODELS

Predictor formula

m_s+n_s sampled points

$$f_k(x) \approx s_k(x) = \sum_{i=1}^{m_s} b_i \psi_i(x) + \sum_{j=1}^{n_s} \beta_j \phi(x - x_j) \quad , \quad k = 1, n_f$$

global basis function

local basis function

kriging model

$$\phi(x - x_j) = e^{-\sum_{j=1}^{n_s} \theta_j |x - x_j|^{p_j}} \qquad \theta_j \geq 0 \qquad p_j \in]0,2]$$

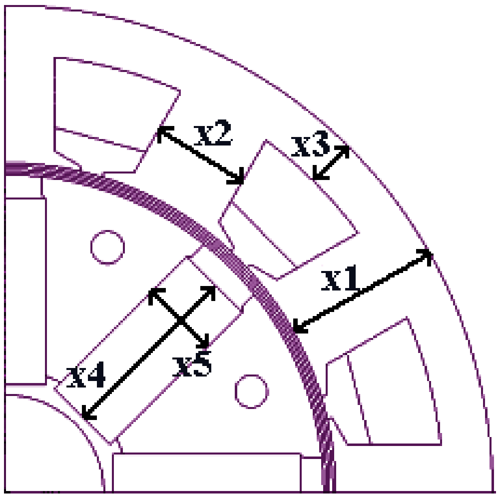
Scalarizing methods

Combine the surrogates of multiple objectives into a preference function; then, single-objective optimisation.

Non-scalarizing methods

consider the surrogate of each objective individually; then, non-dominated solutions.

CASE STUDY



Permanent-magnet generator for automotive applications.

A very similar device was used as the alternator on board of fast cars for sport competitions.

Design problem: identify the shape of the device such that

- power loss in copper windings $f_1(x) = \int_{\Omega_1(x)} \rho [J(x)]^2 \, d\Omega$
- power loss in the iron core $f_2(x) = \int_{\Omega_2(x)} p(B(x)) d\Omega$

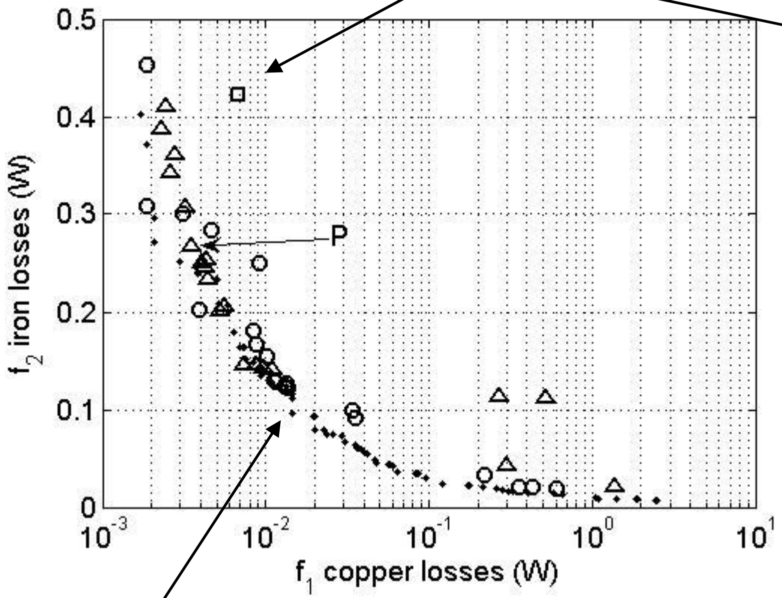
Ω_1 copper volume
Ω_2 iron volume

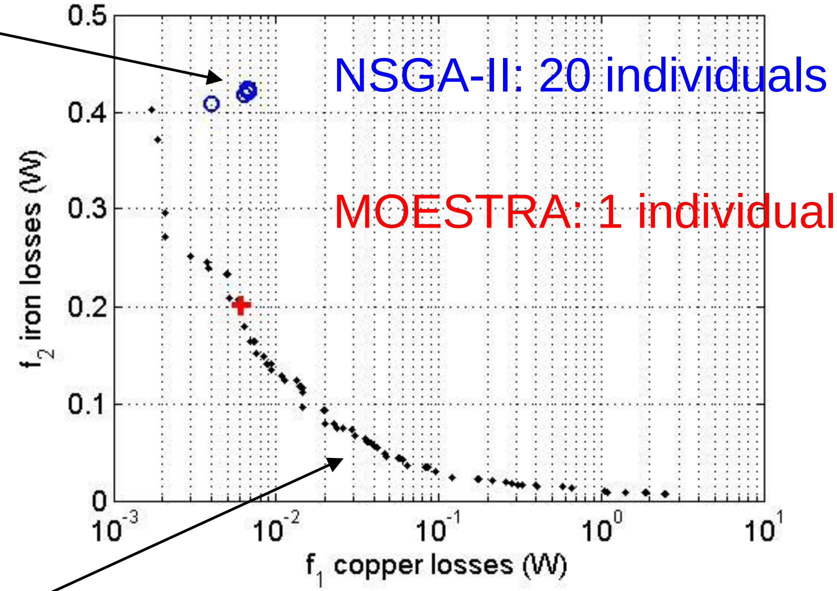
are minimum.

Constraint : load 500 W, no-load peak voltage 50 V , speed 9,000 rpm

NSGA-II AND MOESTRA IN ACTION

prototype



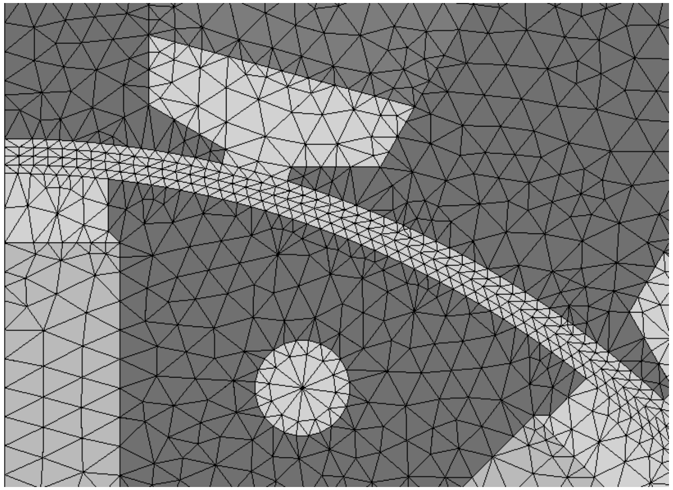


NSGA-II, after 10 gen.s
(2 runs, 20+20 ind.s)

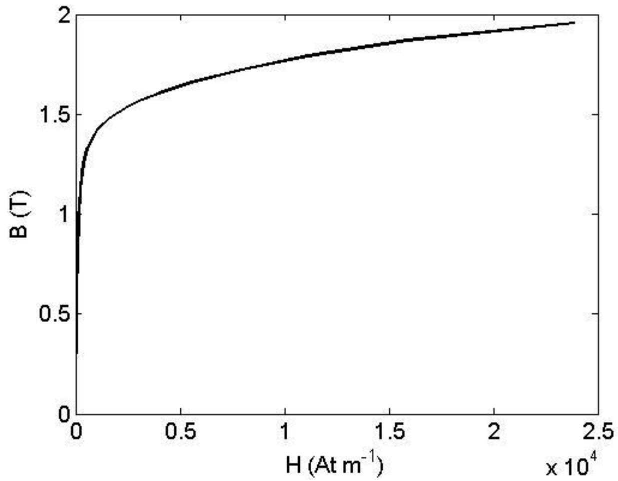
NSGA-II and MOESTRA, after 6,300 s

sampled Pareto front

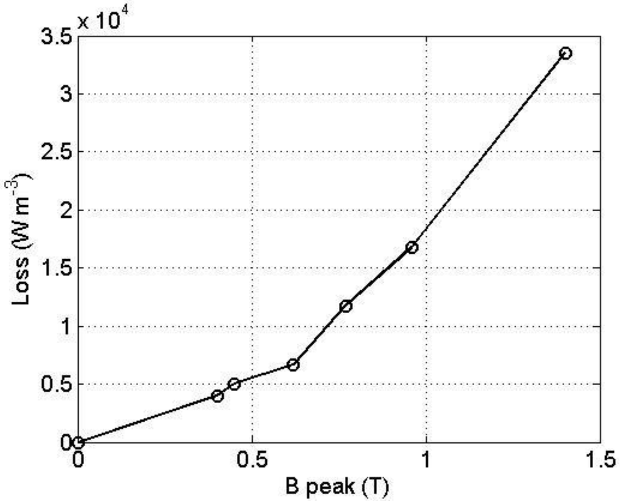
P. Di Barba and M. E. Mognaschi, “Industrial Design with Multiple Criteria: Shape Optimization of a Permanent-Magnet Generator”, T-MAG, vol.45, 2009



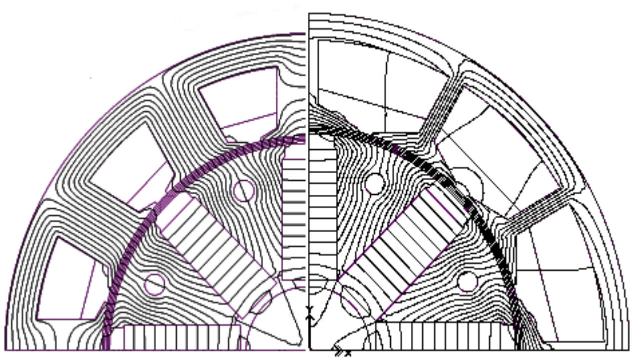
Detail of the FE mesh.



Magnetization curve of iron core.



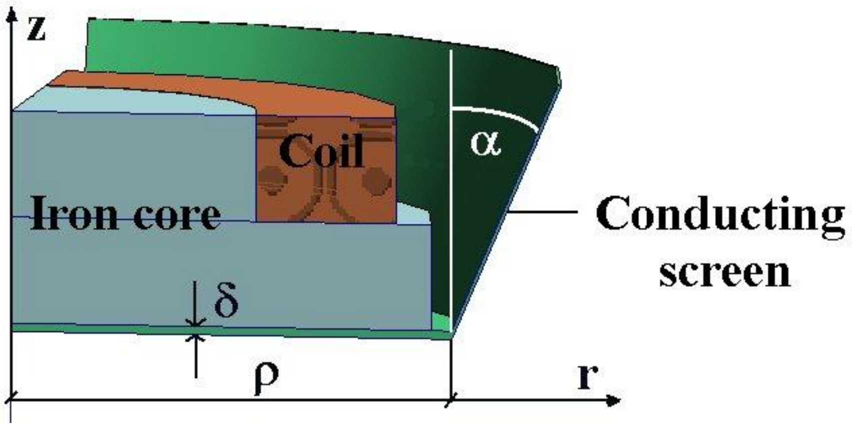
Iron specific-loss curve.

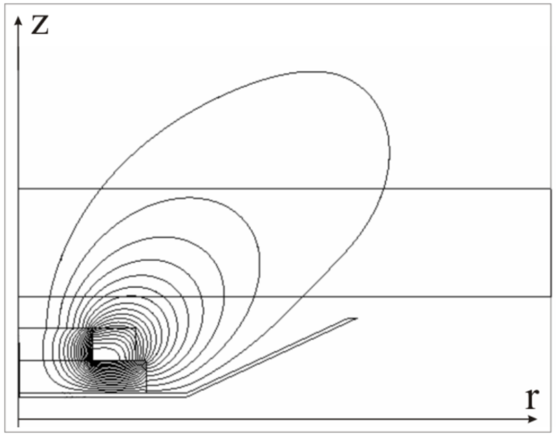


Left: prototype solution, right: a Pareto optimal solution.

NON-CONFLICTING MULTIPLE OBJECTIVES

An axisymmetric antenna
for magnetic induction tomography





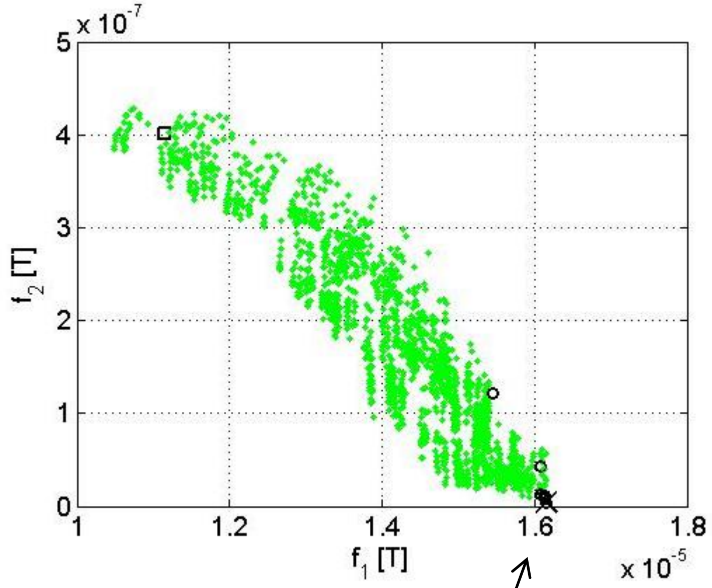
Frequency dependent field

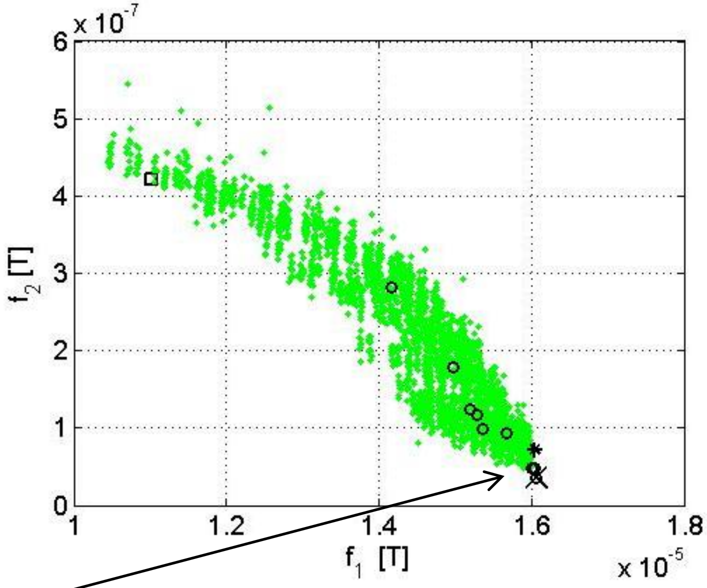
Optimal design problem
Find the antenna shape, identified by variables (α,δ,ρ), such that:
the magnetic field along the antenna axis ($z>0$) is maximum, and simultaneously
the stray field behind the antenna ($z<0$) is minimum.

NON-CONFLICTING MULTIPLE OBJECTIVES (II)

Optimisation results for f = 10 kHz

Optimisation results for f = 100 kHz





The optimum is unique (zero-dimensional Pareto front)

HIGHER-ORDER DIMENSIONALITY n_f>2 (I)

Method of orthogonal projections

Design points are mapped in all possible 2D subspaces

$$\left(f_i, f_j\right), \quad i, j = 1, n_f \quad , \quad i \neq j$$

Effective for objective space representation

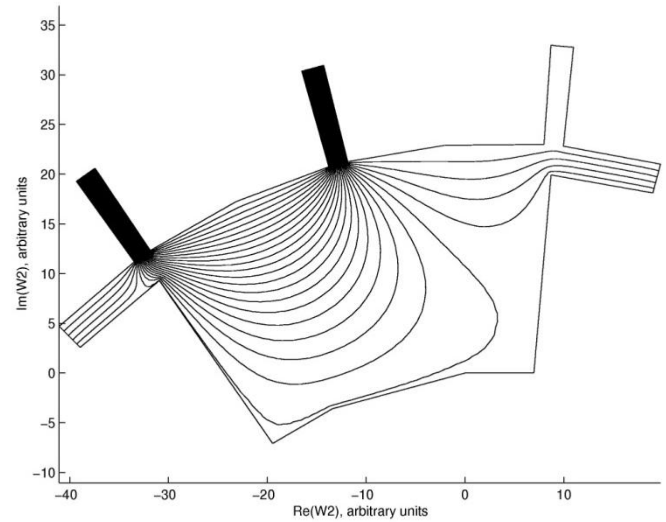
Unpractical for identifying P-optimal solutions

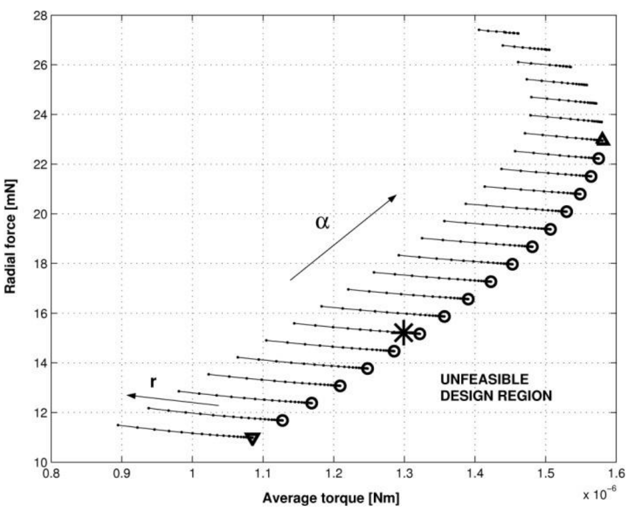
HIGHER-ORDER DIMENSIONALITY $n_f>2$ (II)

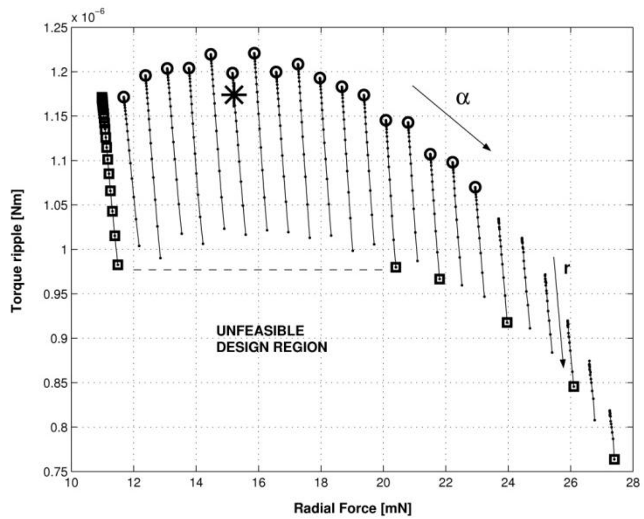
The device
Electrostatic microactuator

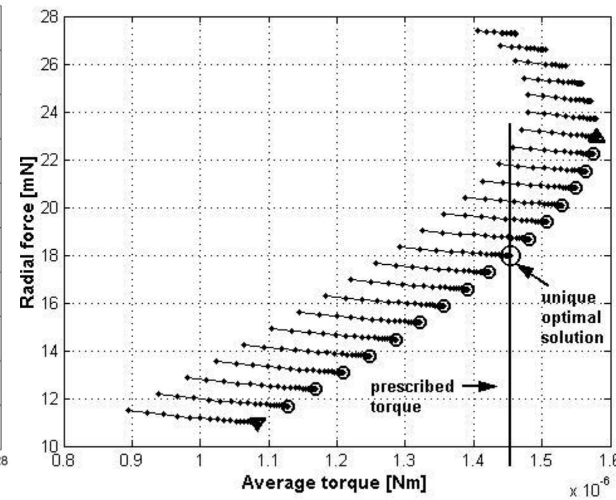
Design variables
rotor inner radius
rotor slot width

Objectives
static torque, to be max
torque ripple, to be min
radial force (friction), to be min









E. Costamagna, P. Di Barba, A. Savini, *Shape design of a MEMS device by Schwarz-Christoffel numerical inversion and Pareto optimality*, COMPEL, vol. 27, 2008

OPEN PROBLEMS IN EMO: BENCHMARKING

In EMO, evaluating the performance of an optimisation algorithm and assessing results is a challenging task.

Some goals of benchmarking :

- Move from test problems to industrial benchmarks
- Investigate topological properties of the PF (convex/non-convex, connected/non-connected, uniformly/non-uniformly spaced)
- Define suitable metrics to measure the distance of a given solution point from the front

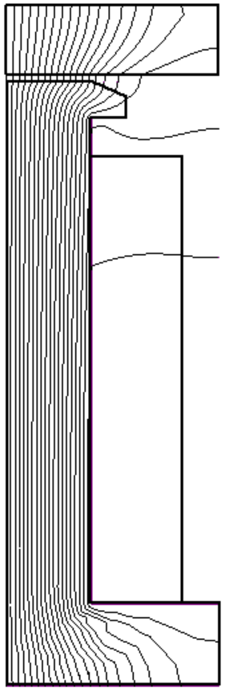
OPEN PROBLEMS IN EMO : BENCHMARKING (II)

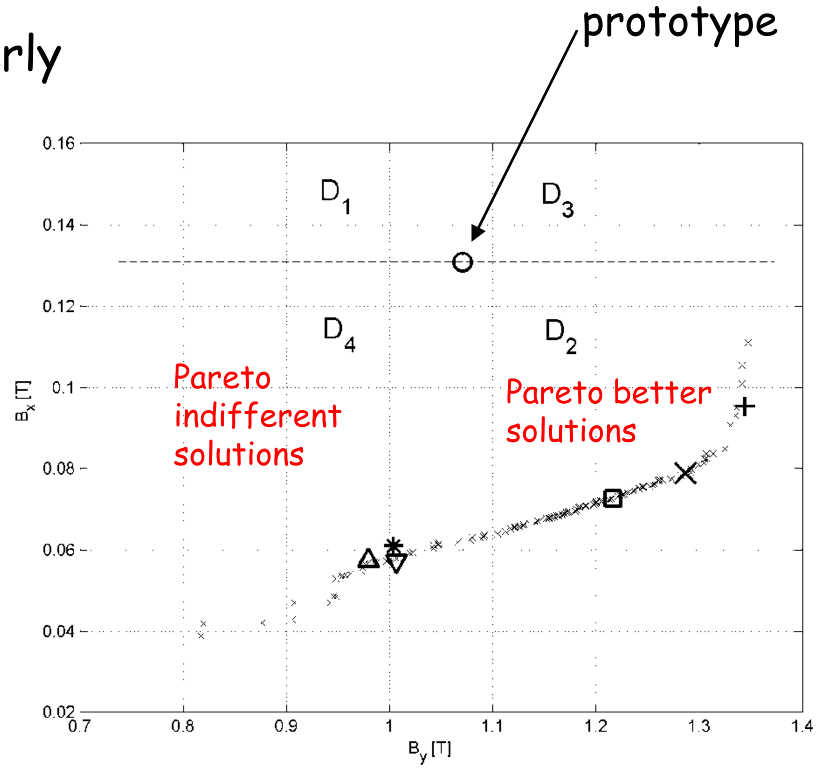
•Handle non-comparable solutions properly

Shape design of a magnetic pole

maximise B_y in the air gap

minimise B_x in the winding

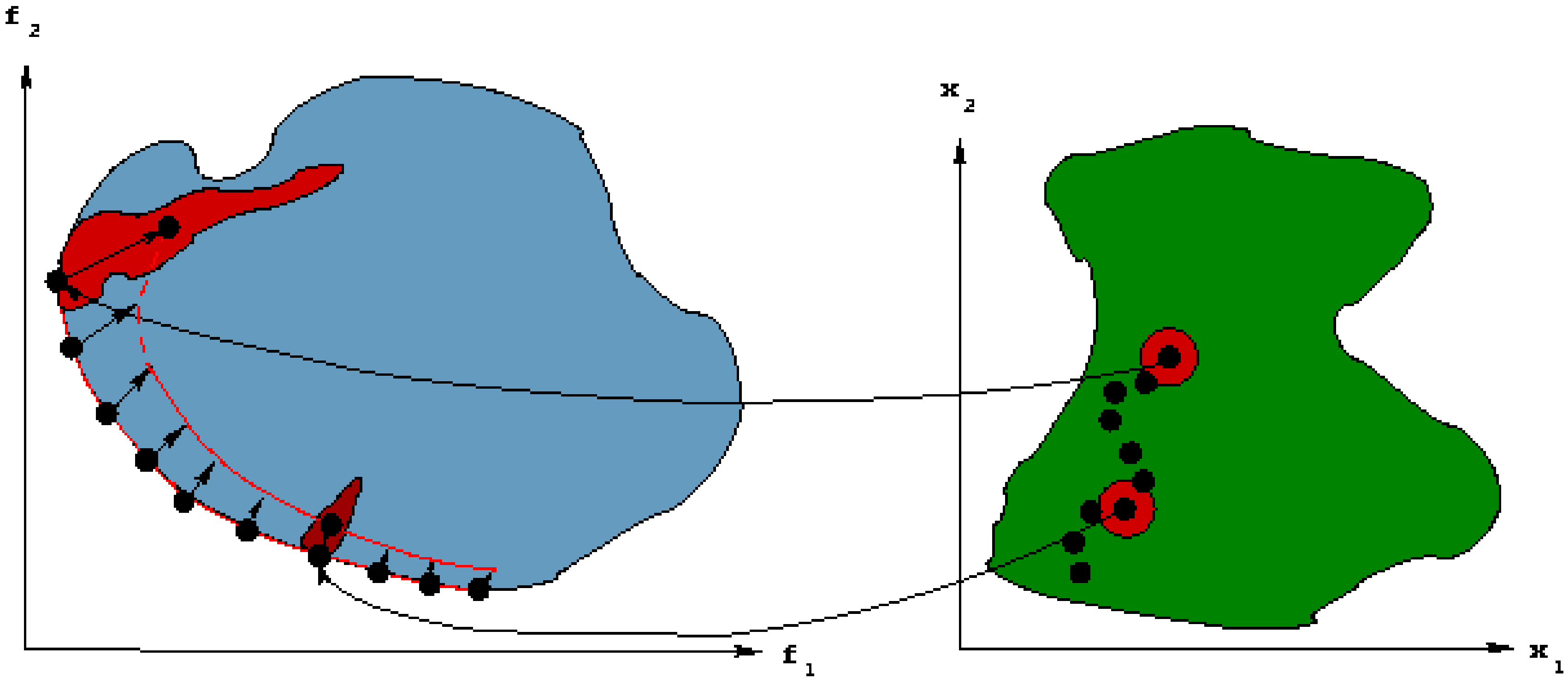




Possibly, the multiobjective formulation of TEAM problems 22 and 25 should be improved.

DESIGN SENSITIVITY AND MOSD

Evaluate the sensitivity of a solution in the objective space (especially, along the PF) with respect to a perturbation in the design space.



DESIGN SENSITIVITY AND MOSD

Numerically derived sensitivity

solution distance $$d(g_i,g_j)=\left[\sum_{k=1}^{n_v}[g_i(k)-g_j(k)]^2\right]^{\frac{1}{2}} \ , \ i=1,n_p-1 \ , \ j=2,n_p \ , \ j>i$$

n_v-dimensional hypercube
encapsulating design space Ω $$V=\prod_{k=1}^{n_v}\left[\sup_{\Omega}\|g(k)\|-\inf_{\Omega}\|g(k)\|\right]$$

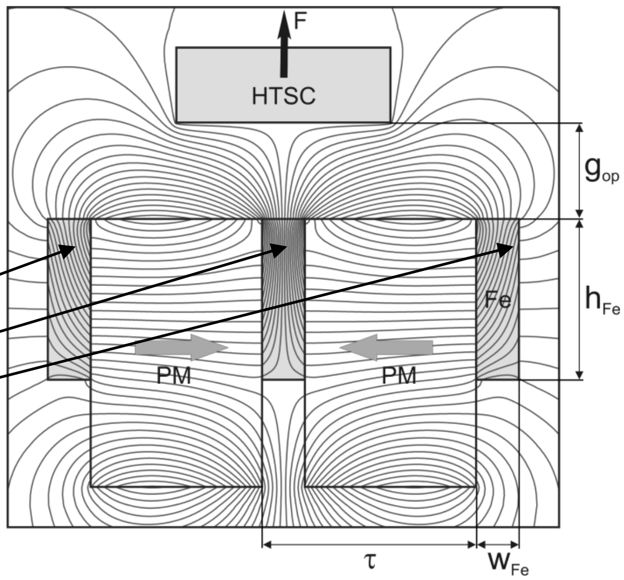
distance threshold $$\delta=\sqrt{n_v}\left[(n_p)^{-1}V\right]^{\frac{1}{n_v}}$$ (e.g. hypercube diagonal)

$d(\tilde{g},g_j)<\delta \ , \ j=2,n_p$ ⇒ perturbation domain ω relevant to $\tilde{g}$

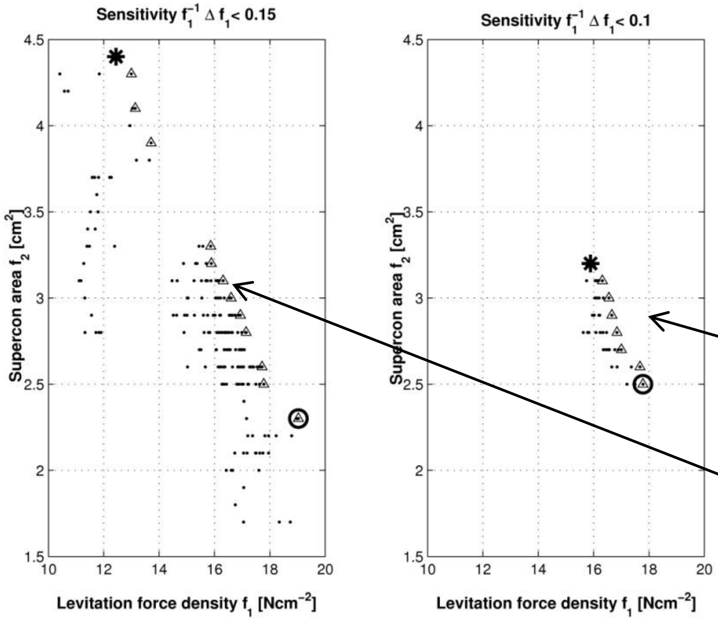
sensitivity $$s(\tilde{g})=[f_\ell(\tilde{g})]^{-1}\underbrace{\left[\sup_{\omega} f_\ell(g)-\inf_{\omega} f_\ell(g)\right]}_{\text{discrete Lipschitz's constant}}\equiv\frac{\Delta f_\ell}{f_\ell} \ , \ f_\ell(\tilde{g})\neq 0 \ , \ \ell=1,n_\ell$$

PERFORMANCE *vs* SENSITIVITY

A *maglev* device



Design variables
dimensions of permanent magnets and field correctors



Dependence of Pareto front on force sensitivity *s*

s<0.1

s<0.15

Objectives
- levitation force
- supercon area

A multi-scale evolutionary search

The adaption rate $0<\lambda<1$ of the FE mesh is ruled by the annealing operator of a basic evolution strategy.

A low-cost mesh is generated when a large search radius is taken on and, conversely, a finer mesh is generated when a small region is investigated.

$$\lambda(k) = \lambda_{min}\left(1 - \frac{m(k)}{n}\right) + \lambda_{max}\frac{m(k)}{n}$$

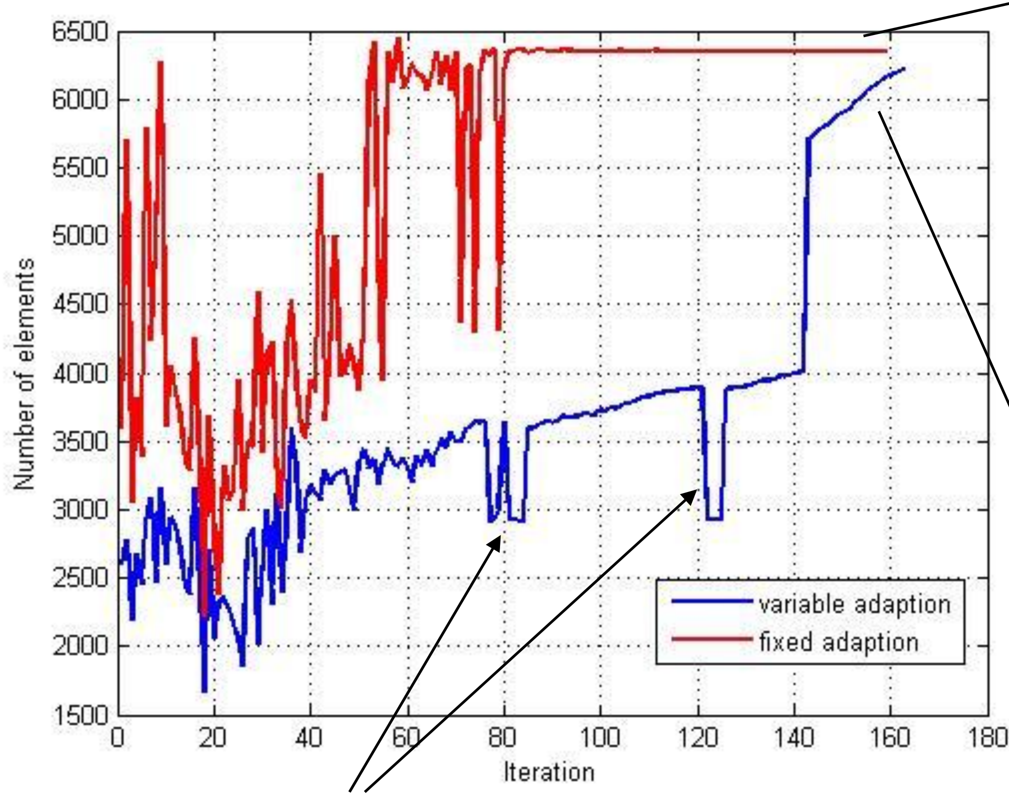
$$n = \frac{\log d_f - \log d_0}{\log q}$$

$$m(k) = \frac{\log d(k) - \log d_0}{\log q}$$

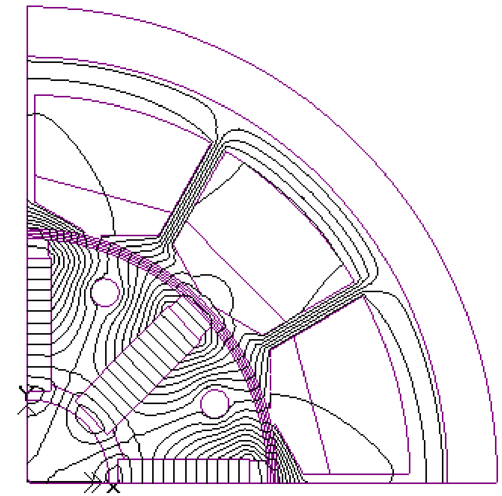
d_0 initial search tolerance
d_f final search tolerance
k iteration index
q annealing rate

CASE STUDY

Stopping criterion: search tolerance < 10^{-6}

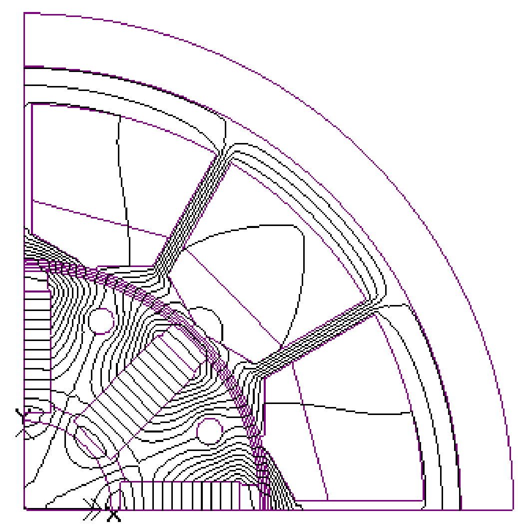


larger search radius



x=[10.33, 1.01, 2.30, 7.99, 2.87] mm

f=[4.55, 160] mW



x=[10.87, 0.97, 2.11, 6.83, 3.05] mm

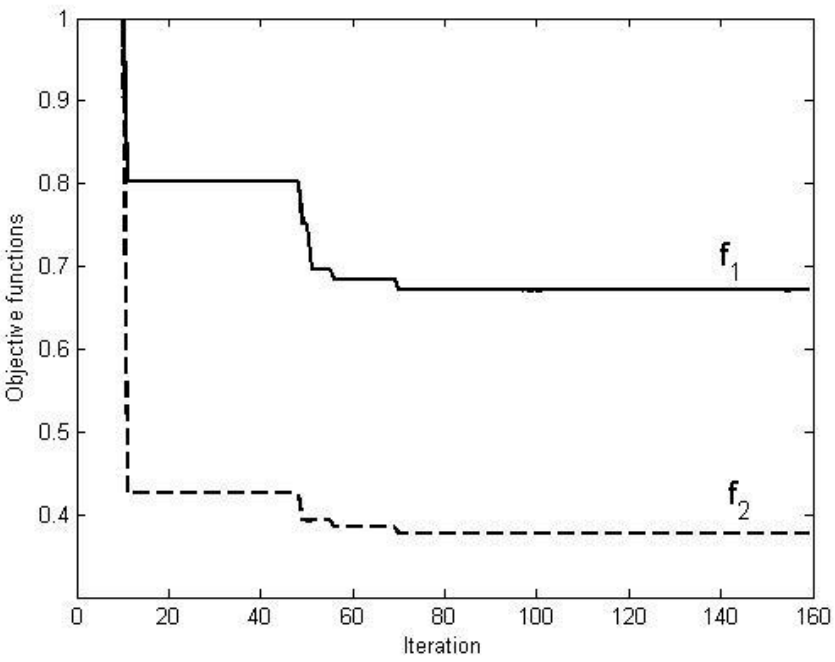
f=[4.36, 150] mW

Fixed *vs* variable adaption

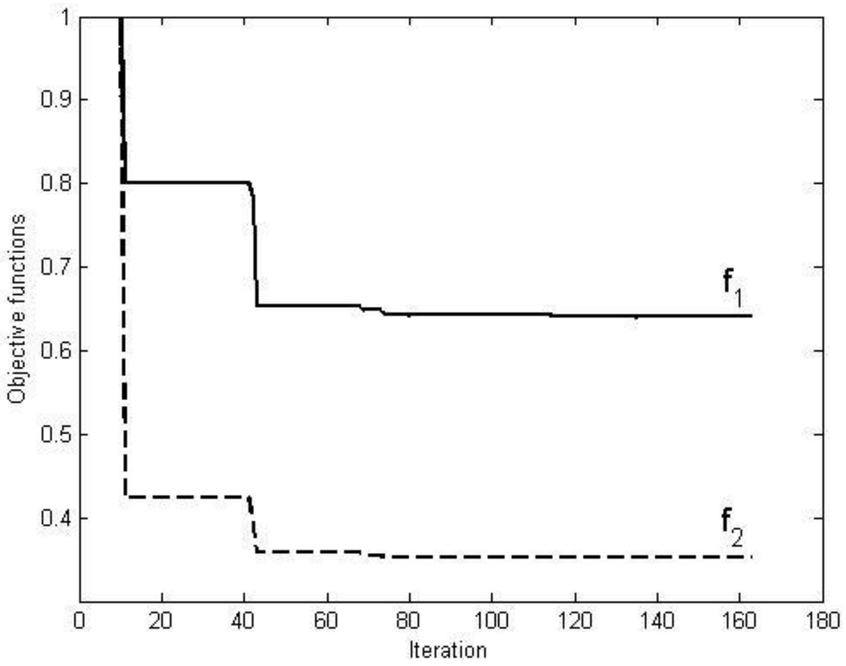
(fixed λ=0.15, variable $0.02<\lambda<0.2$)

CASE STUDY (II)

Objective function history



Fixed adaption

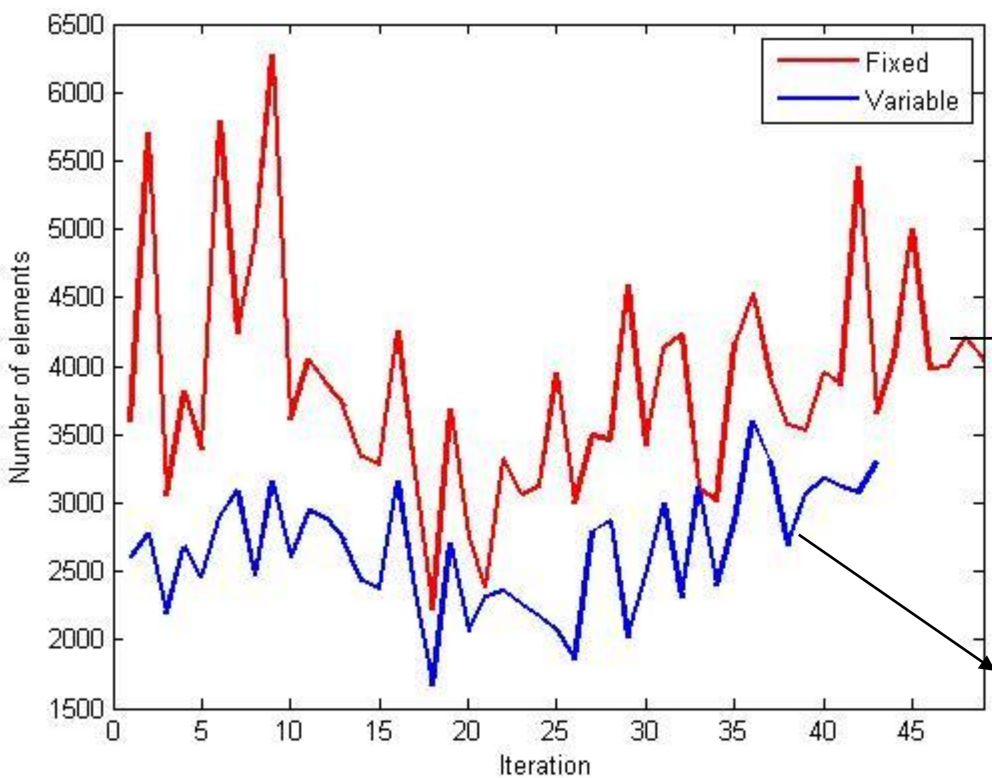


Variable adaption

CASE STUDY (III)

User-defined accuracy: the optimisation stops when

$$\rho_i(k) \equiv \frac{f_{ki}}{f_{0i}} \leq \eta \ , \ i=1,2 \ , \ k \geq 0$$

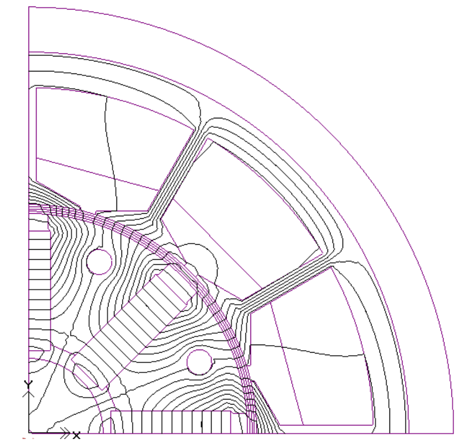


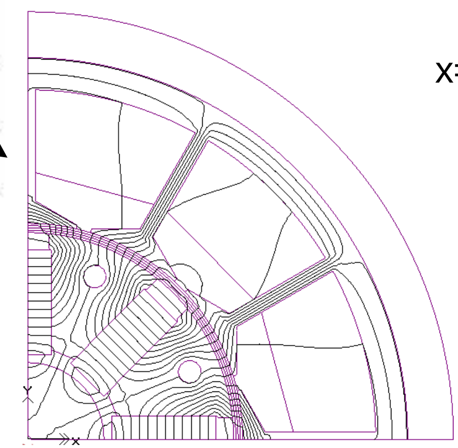
Prescribed η = 0.75

x=[10.18, 1.06, 2.38, 8.03, 2.95] mm

ρ_1=0.75, ρ_2=0.4

k=49





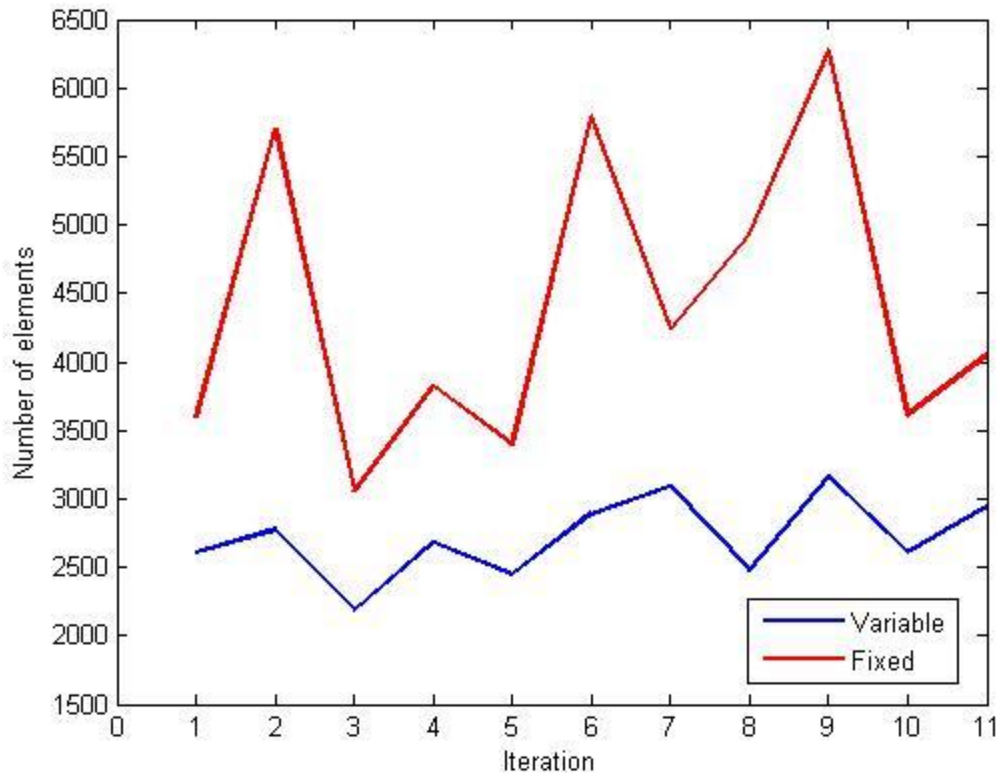
x=[10.81, 0.99, 2.09, 6.84, 3.07] mm

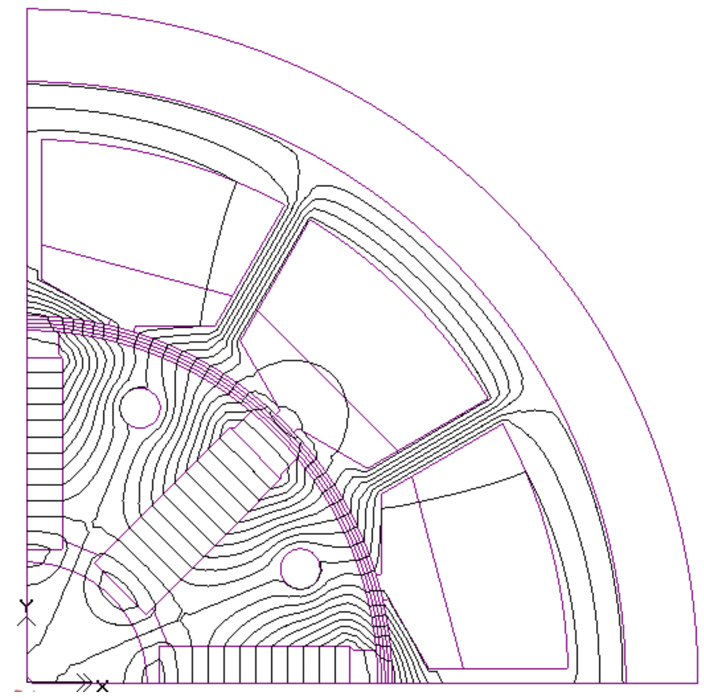
ρ_1=0.65, ρ_2=0.36

k=43

CASE STUDY (IV)

User-defined time: the optimisation stops after e.g. 1 hour





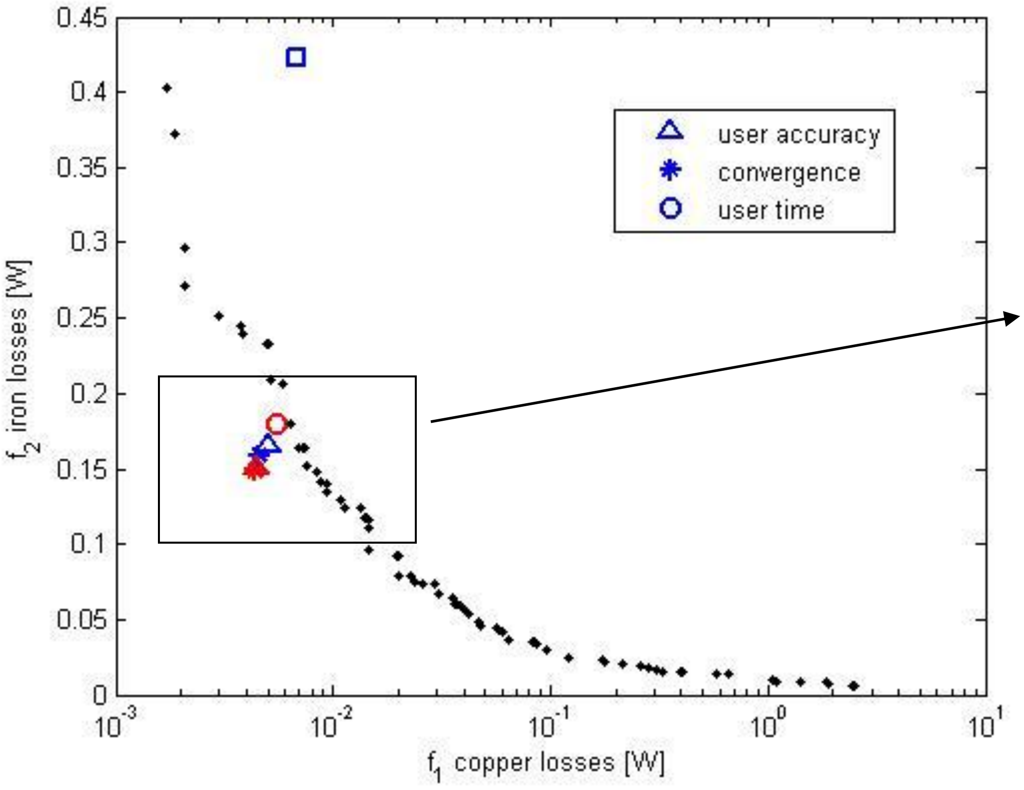
x=[9.70, 1.21, 2.40, 7.93, 3.00] mm

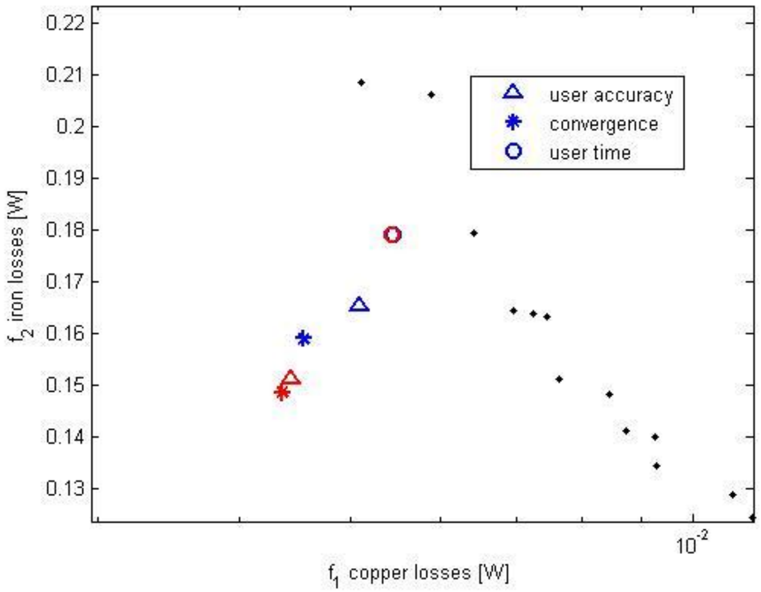
f_1=5.5 mW, f_2=179.3 mW

11 iterations done in 1 hour

Improvement: 20% for f_1 , 60% for f_2

CASE STUDY (V)





Fixed adaption

Variable adaption

AN ALTERNATIVE: NASH GAMES

An *a priori* method to provide the designer with a single optimum.

Each player minimises his own objective by varying a single variable and assuming that the values of the remaining n-1 objectives are fixed by the other n-1 players. If it happens that no player can further reduce his objective, it means that the system has converged to an equilibrium.

Let Ω and Ω_i be the global design space and the design space of the i-th objective such that $\Omega_i \subset \Omega = \Omega_1 \times \ldots \times \Omega_i \times \ldots \times \Omega_n$

Then, a point $(\tilde{x}_1, \ldots, \tilde{x}_i, \ldots, \tilde{x}_n) \in \Omega$ is a Nash equilibrium (NE) if

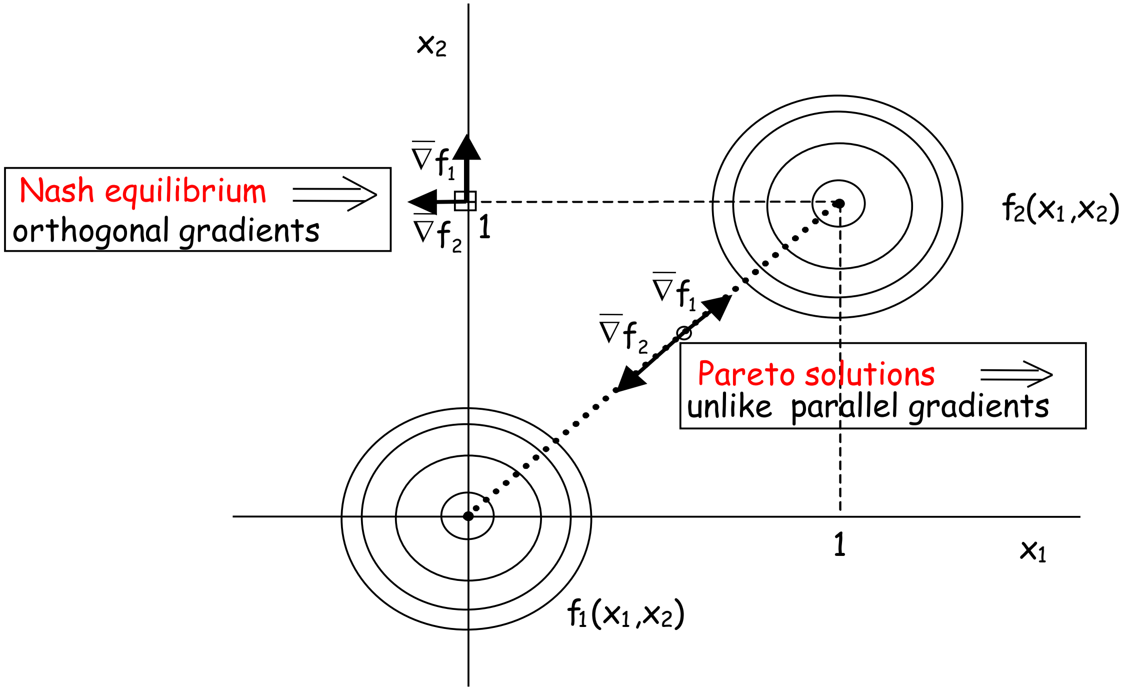
$$f_i(\tilde{x}_1, \ldots, \tilde{x}_{i-1}, \tilde{x}_i, \tilde{x}_{i+1}, \ldots, \tilde{x}_n) =$$

$$= \inf_{x_i \in \Omega_i} f_i(\tilde{x}_1, \ldots, \tilde{x}_{i-1}, x_i, \tilde{x}_{i+1}, \ldots, \tilde{x}_n) \quad \forall i = 1, \ldots, n$$

AN ALTERNATIVE: NASH GAMES

2D analytical
test case

$$f_1 = (x_1^2 + x_2^2)$$
$$f_2 = (x_1 - 1)^2 + (x_2 - 1)^2$$

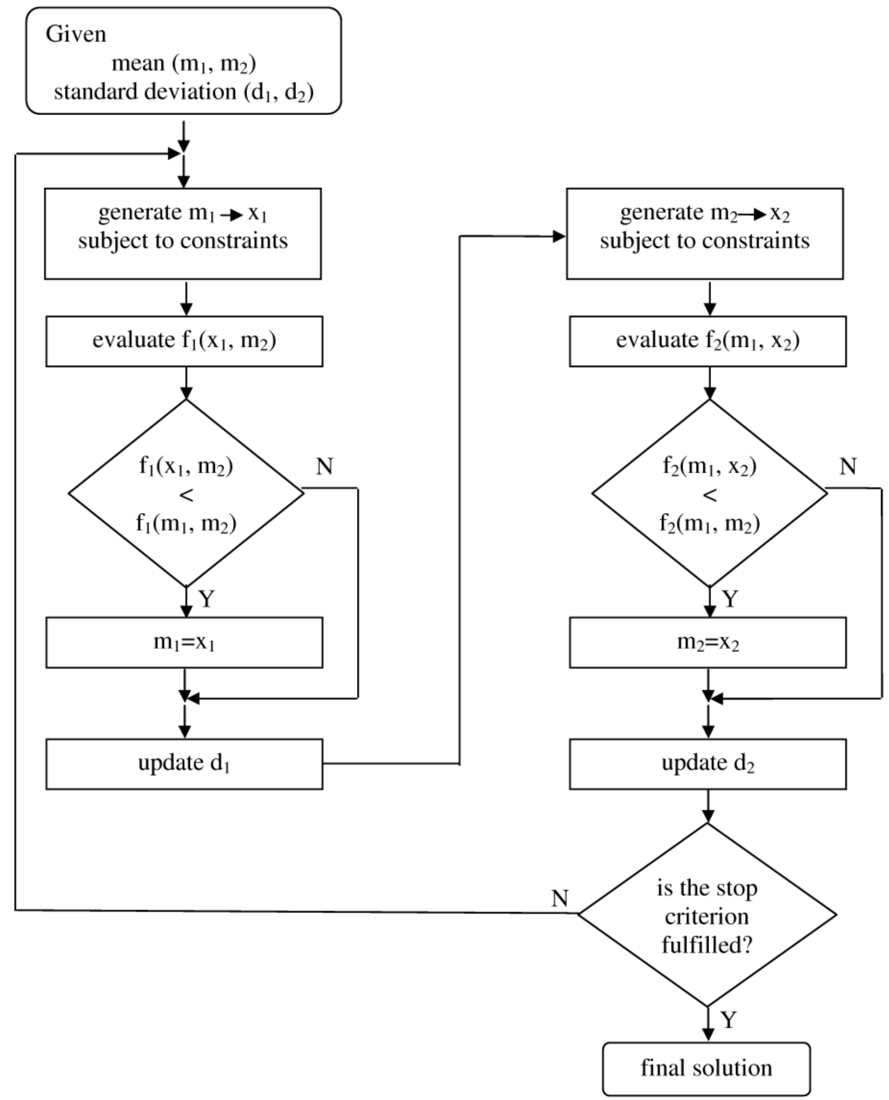


NASH GAMES: NUMERICAL IMPLEMENTATION

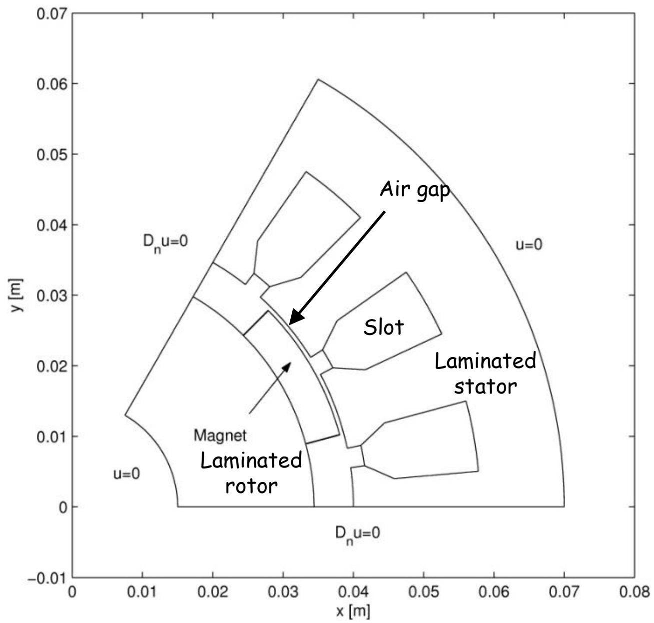
Player 1 optimizes $f_1(x_1,x_2)$ acting on x_1 and receiving x_2 from player 2 at the previous iteration; then, player 1 sends the result to player 2.

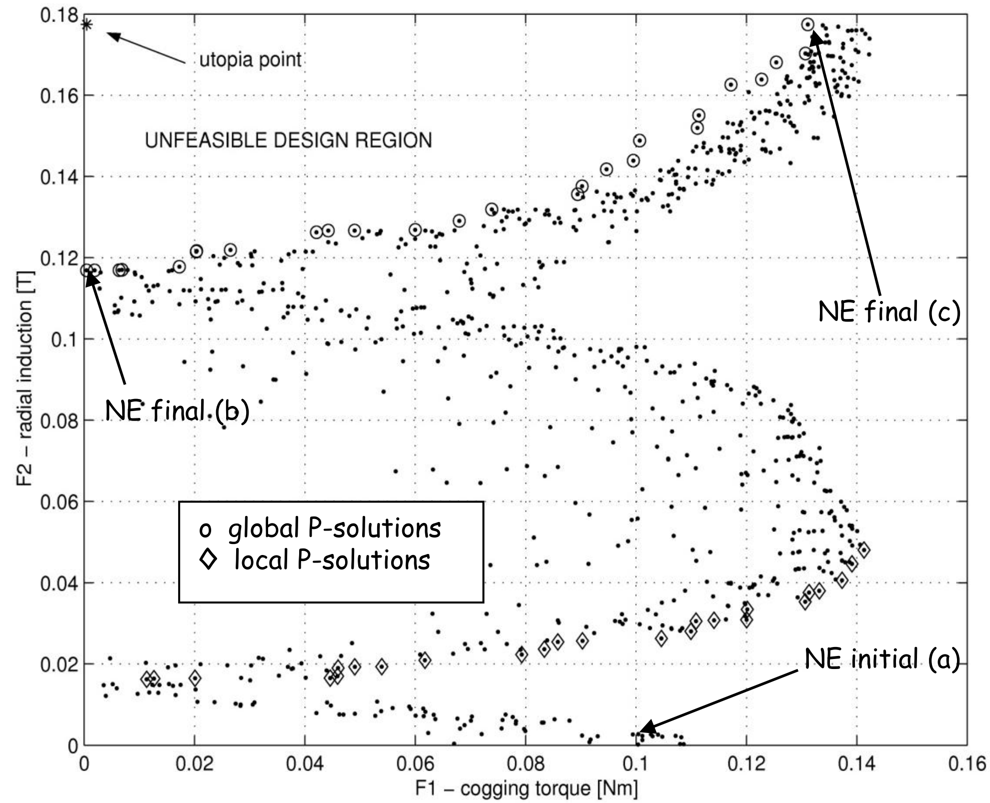
Player 2 optimizes $f_2(x_1,x_2)$ acting on x_2 and receiving x_1 from player 2 at the previous iteration; then, player 2 sends the result to player 1.

The game is over (Nash equilibrium) when neither player 1 nor player 2 can further improve their objectives.



PM THREE-PHASE MOTOR





Design variables :
height and width of magnet

Objectives for no-load operation:
cogging torque (to be min), air-gap radial induction (to be max)

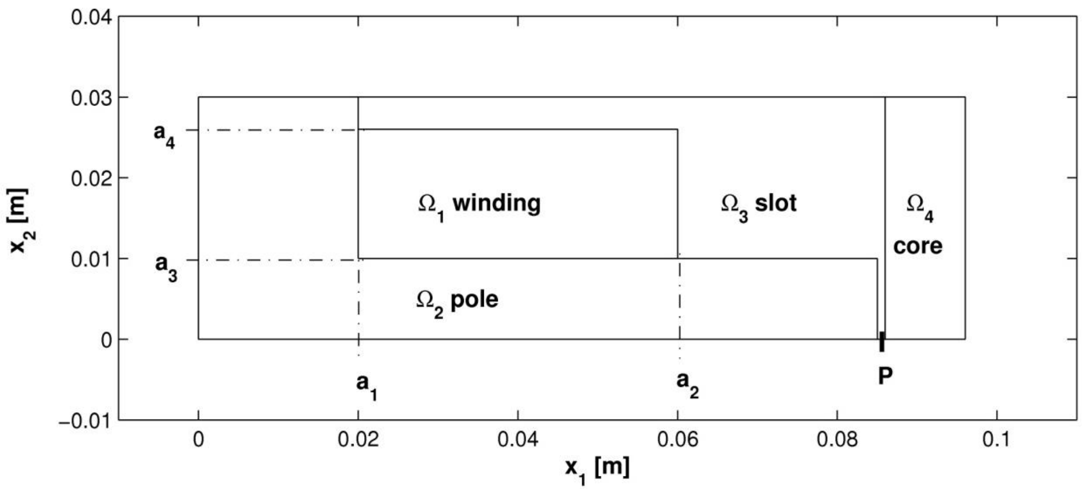
P. Di Barba, *Strategies of game theory for the automated optimal design in electromechanics*, Intl J of Applied Electromagnetics and Mechanics 27 (2008) 1-21

FROM STATIC TO DYNAMIC PARETO FRONTS

Shape design of a magnetic pole

Design variables :

(a_1,a_2,a_3,a_4)



The problem reads: find the time-dependent family of non-dominated solutions from $t = 0^+$ to steady state such that

• air-gap induction is maximum

• power loss in the winding is minimum

under the constraint that

the power loss in the pole and the core at a given time instant ($t=10^{-2}\tau$) is not greater than the power loss in the winding.

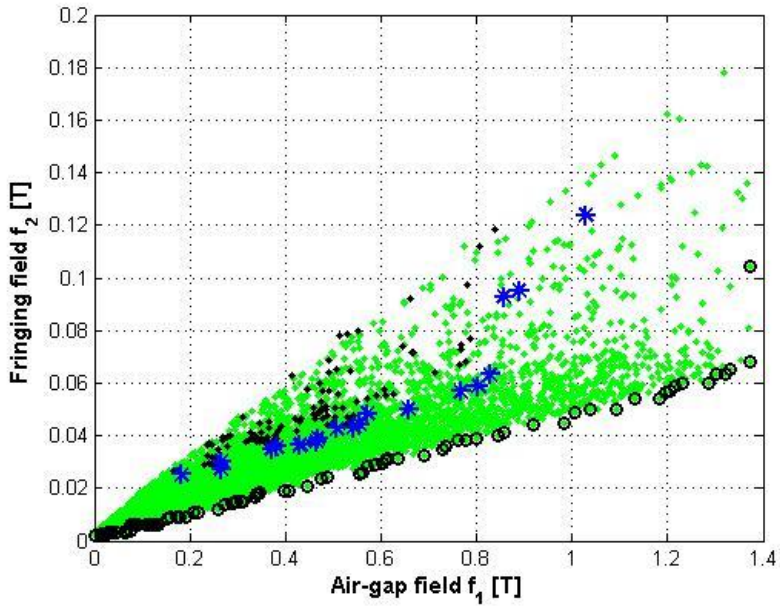
Time constant depends on geometry

$$\tau_1 = \mu\sigma_2\left[\inf_k \lambda(k)\right]^2 \quad , \quad k = 1, n_p$$

$$\lambda(k) = \min[a_2(k) - a_1(k), a_4(k) - a_3(k)] \quad , \quad k = 1, n_p$$

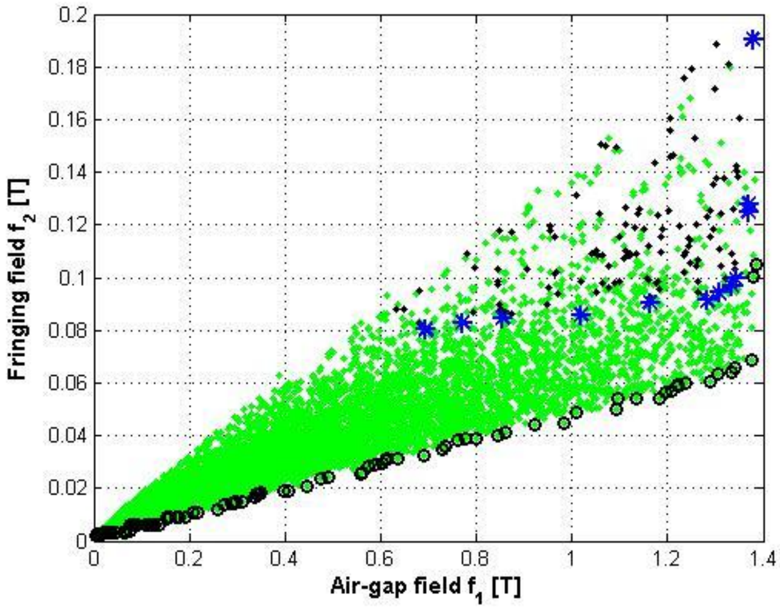
FROM STATIC TO DYNAMIC PARETO FRONTS (II)

Objective space at t = τ



time-unconstrained (circle) and time-constrained (star) fronts

Objective space at steady state



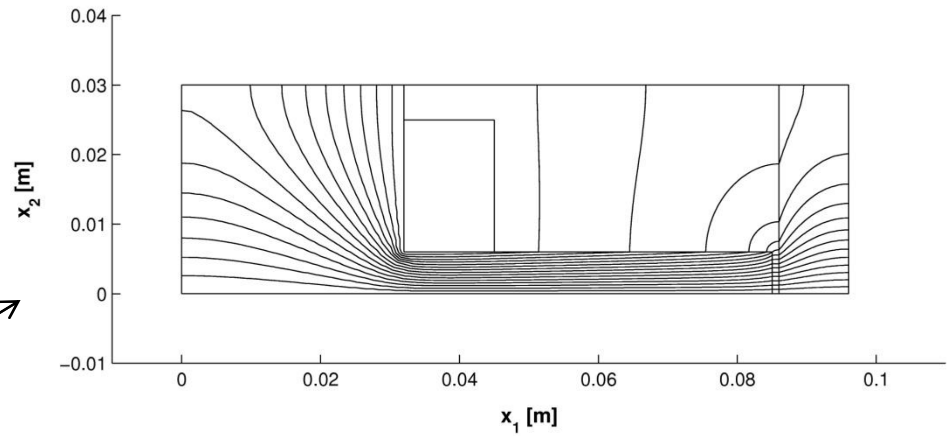
time-unconstrained (circle) and time-constrained (star) fronts

The energy constraint, active in the first part of the transient magnetic diffusion, influences the Pareto front shape at any subsequent time instant !

P. Di Barba, A. Lorenzi, A. Savini, *Dynamic Pareto fronts and optimal control of geometry in a problem of transient magnetic diffusion*, IET Science, Measurement and Technology, 2008, vol.2, no.3, 114-121

FROM STATIC TO DYNAMIC PARETO FRONTS (III)

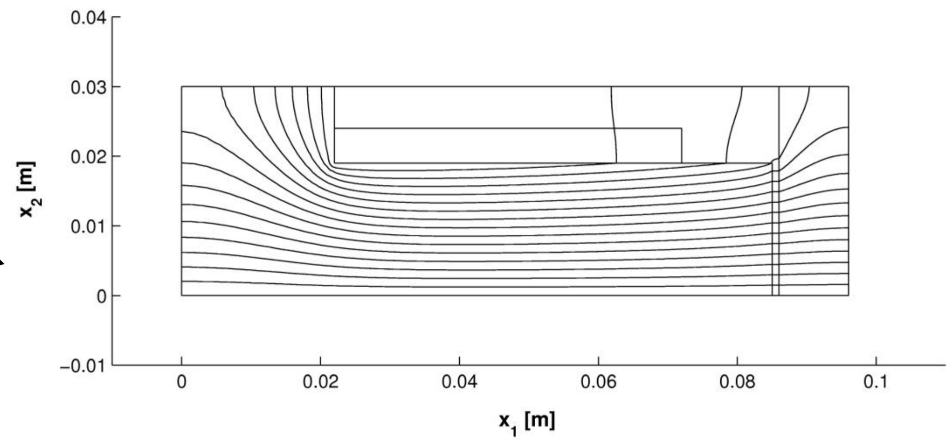
Also the solution shape is different !



Geometry and flux lines of a non-dominated solution at steady state (time-unconstrained PF) :
f_1 = 846.031 mT, f_2 = 40.073 mT (prescribed f_1 = 850 mT); a_1 = 32 mm, a_2 = 45 mm, a_3 = 6 mm, a_4 = 25 mm.

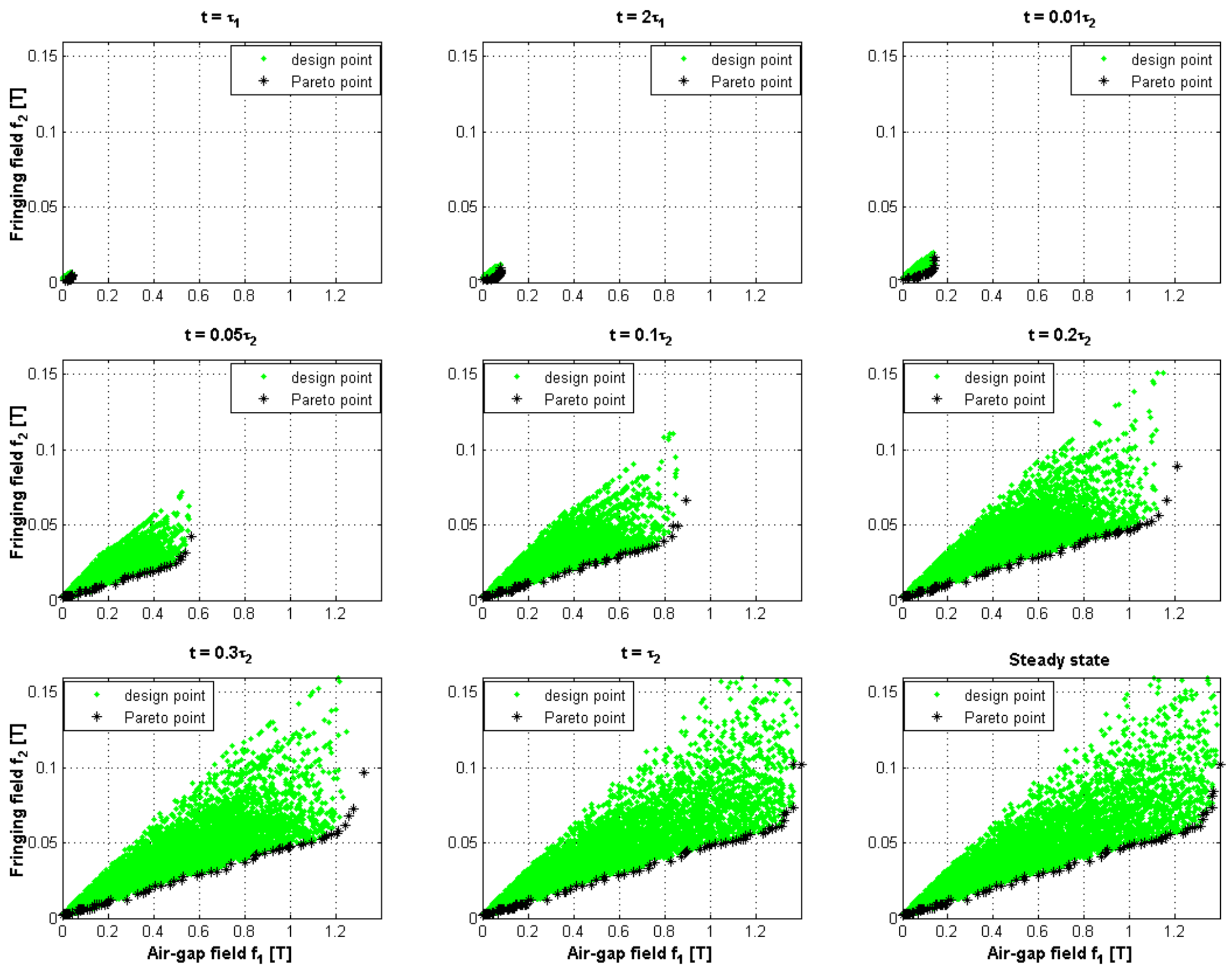
Time unconstrained

Time constrained



Geometry and flux lines of a non-dominated solution at steady state (time-constrained PF) :
f_1 = 855.773 mT, f_2 = 84.628 mT (prescribed f_1 = 850 mT); a_1 = 22 mm, a_2 = 72 mm, a_3 = 19 mm, a_4 = 24 mm.

FROM STATIC TO DYNAMIC PARETO FRONTS (IV)



If the energy constraint is not active, the problem becomes adynamic.

MOVING ALONG THE PARETO FRONT

John necessary condition:

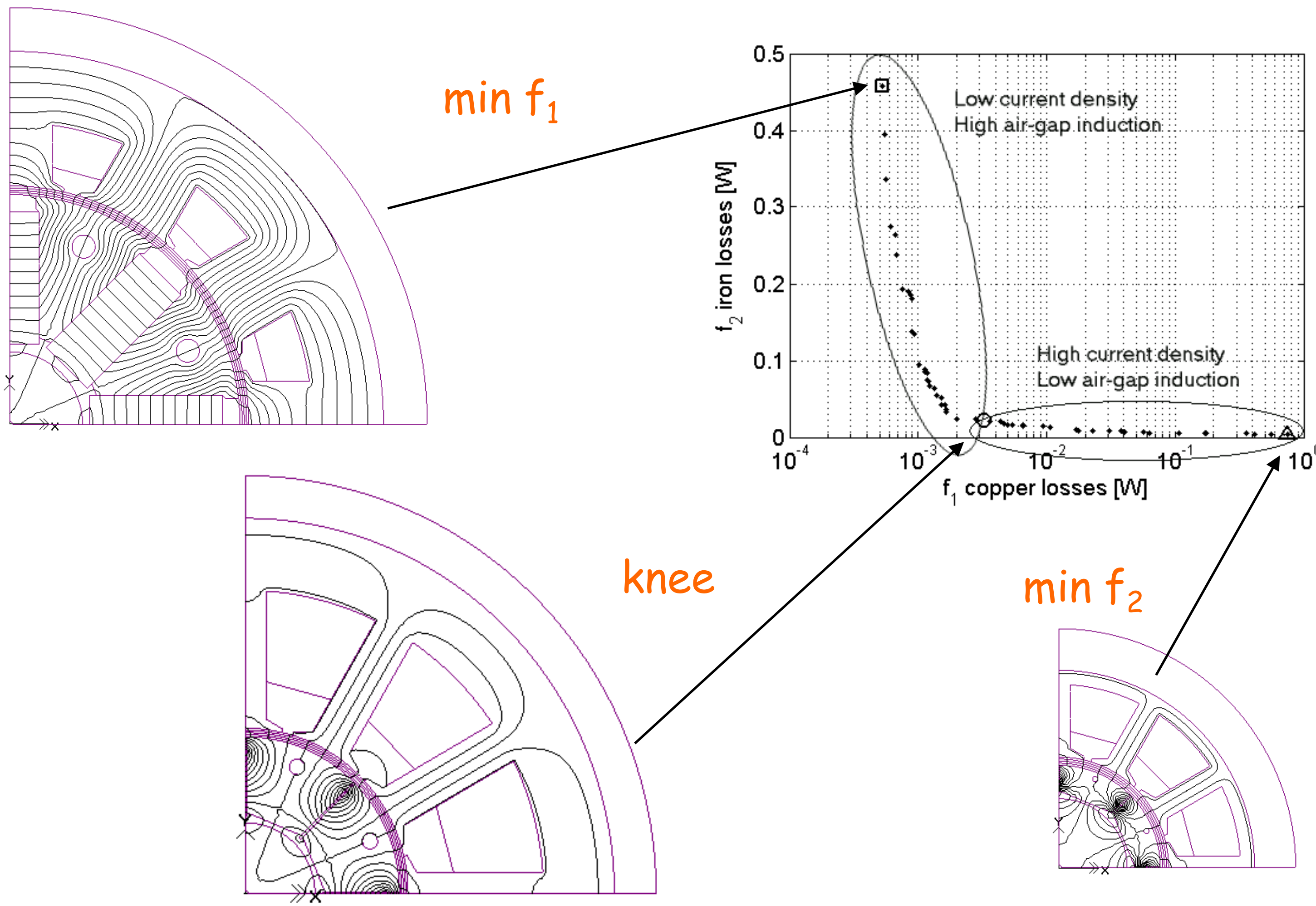
Pareto optimal solution $\tilde{x}$ satisfies

1. $$\sum_{i=1}^{n_f} \lambda_i \overline{\nabla} f_i(\tilde{x}) = \sum_{k=1}^{n_c} \ell_k \overline{\nabla} g_k(\tilde{x})$$ and

2. $$\ell_k g_k(\tilde{x}) = 0 \;,\; k = 1, n_c$$

- Requires differentiable objectives and constraints
- Outlines the existence of some common properties among Pareto-optimal solutions
- If objectives $f_i(x)$, $i=1,n_f$ are convex and Ω is a convex region, the condition is sufficient too.

MOVING ALONG THE PARETO FRONT (II)



P. Di Barba, M.E. Mognaschi, *Sorting Pareto solutions: a principle of optimal design for electrical machines*, COMPEL, vol. 28, 2009

BEYOND EMO

Evolutionary, genetic and migratory algorithms, often employed in MOO, are powerful, but affected by some inherent limits, the most evident of which is the absence of theoretical proofs of convergence.

BEYOND EMO (II)

Individuals of a population-based method of optimisation run towards improvement through a randomness guided by a set of possible heuristics.

An alternative way is developing a statistical method to identify the regions of the X space – the most interesting one to the designer – which are more likely to map onto P-optimal solutions.

The designer, then, should be provided not with a large collection of supposed-optimal individuals, but with a distribution of probability in the X space, which yields optimal configurations with a given degree of certainty.

BEYOND EMO (III)

A formulation of a MOO problem could rely on the Bayes theorem, the goal being just shaping some probability surfaces, to identify the most promising candidate regions for P-optimal solutions.

The problem is no more in terms of an evolving population of individuals, but covering the search space with a probability density, to eventually know what subsets are likely to be a part of the PS.

BEYOND EMO (IV)

Let an optimisation process have already produced some individuals, among which the non-dominated ones have been ranked out.

Then, given a point belonging to the X space, its probability of belonging to the P-set is proportional to its probability of mapping onto a non-dominated point in the Y space, times the probability that a non-dominated point be P-optimal.

BAYESIAN IMAGING AND MO PROBLEMS

Given the *a priori* information I
and defined the two propositions:

$\mu(x)\ ,\ x \in X$ ⟺ "x belongs to the PS"

$\phi(y)\ ,\ y \in Y$ ⟺ "y belongs to the PF"

the Bayes theorem reads

backward mapping term

$$p\left(\phi(y)\middle|\,\mu(x),I\right)=\frac{\overbrace{p\left(\mu(x)\middle|\,\phi(y),I\right)}\;\overbrace{p\left(\phi(y)\middle|\,I\right)}}{p\left(\mu(x)\middle|\,I\right)}$$

forward mapping term

stopping term: the probability for any point y to belong to the PF

normalizing constant

BAYESIAN IMAGING AND MO PROBLEMS
SHAPE DESIGN OF A LINEAR ACTUATOR

Design vector $a = (h_1, h_2, \ell, x, \alpha) \in \Omega \subset \Re^5$

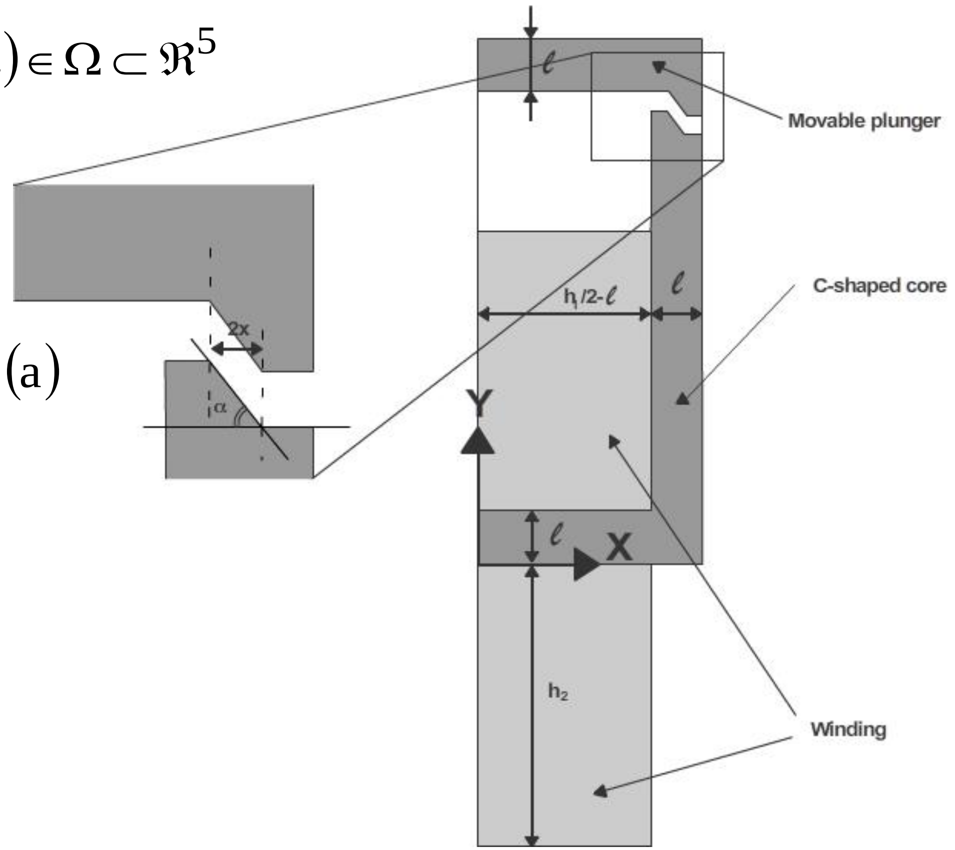
Find $\inf\limits_{a \in \Omega} C(a)$

$$C(a) = c_{iron} V_{iron}(a) + c_{copper} V_{copper}(a)$$

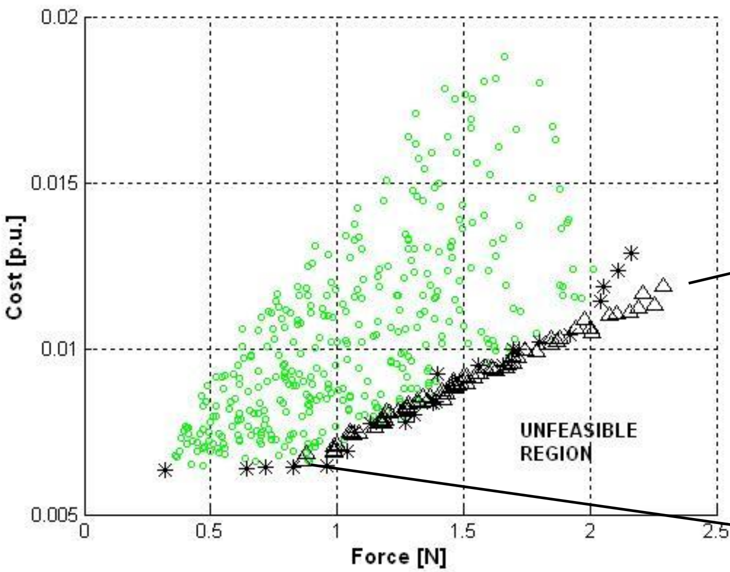
and $\sup\limits_{a \in \Omega} F_y(a)$

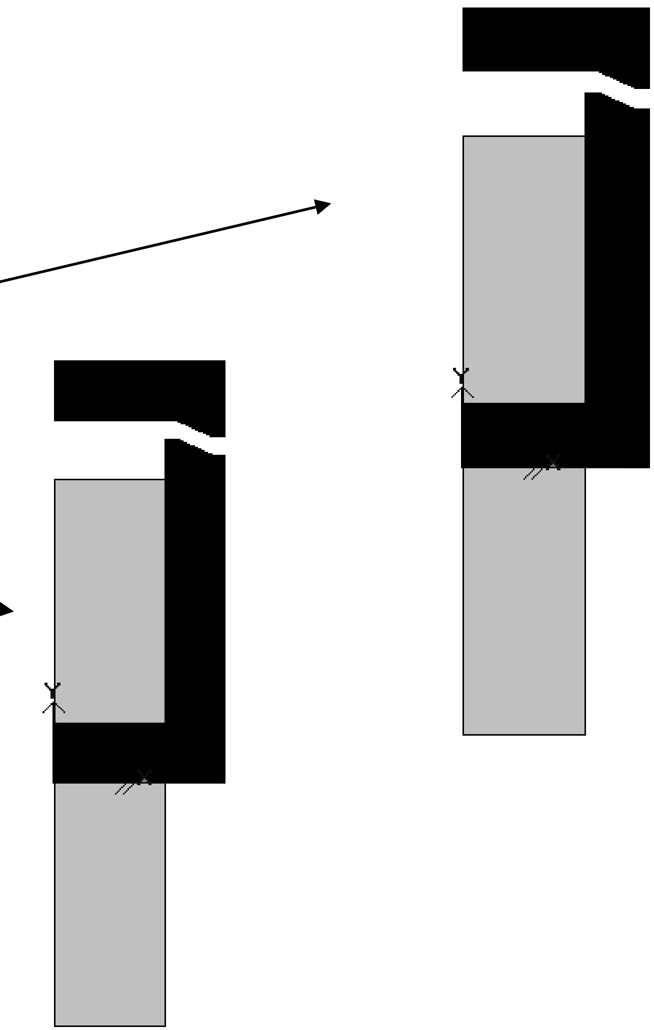
$$F_y(a) \approx \left. \frac{\Delta W'(a)}{\Delta y} \right|_{g=g_0}$$

subject to $\sup\limits_{\Omega_w(a)} |B_y(a)| \le B_0$



OPTIMISATION RESULTS (I)

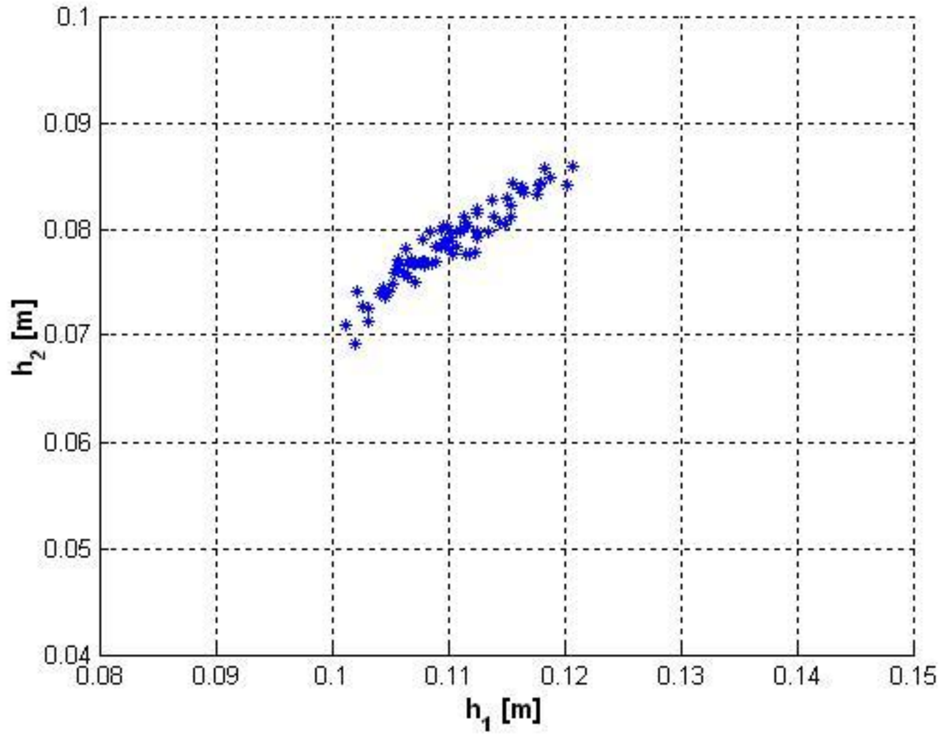




F-space sampling (circle), with relevant PF (star), and PF derived after optimisation (triangle).

OPTIMISATION RESULTS (II)

PS projected on the (h_1,h_2) plane after optimisation



Other optimal solutions can be generated at zero cost, by means of new extractions, until the requirements of the designer in terms of likelihood are met.

Pareto optimality and MEMS design

Fostered by the development of new technologies, micro-electro-mechanical systems (MEMS) are massively present on board of vehicles, within information equipment, in manufacturing systems as well as in medical and healthcare equipment.

The miniaturisation of electromechanical systems will impact our society as deeply as did the mass production of electronic systems in the latest forty.

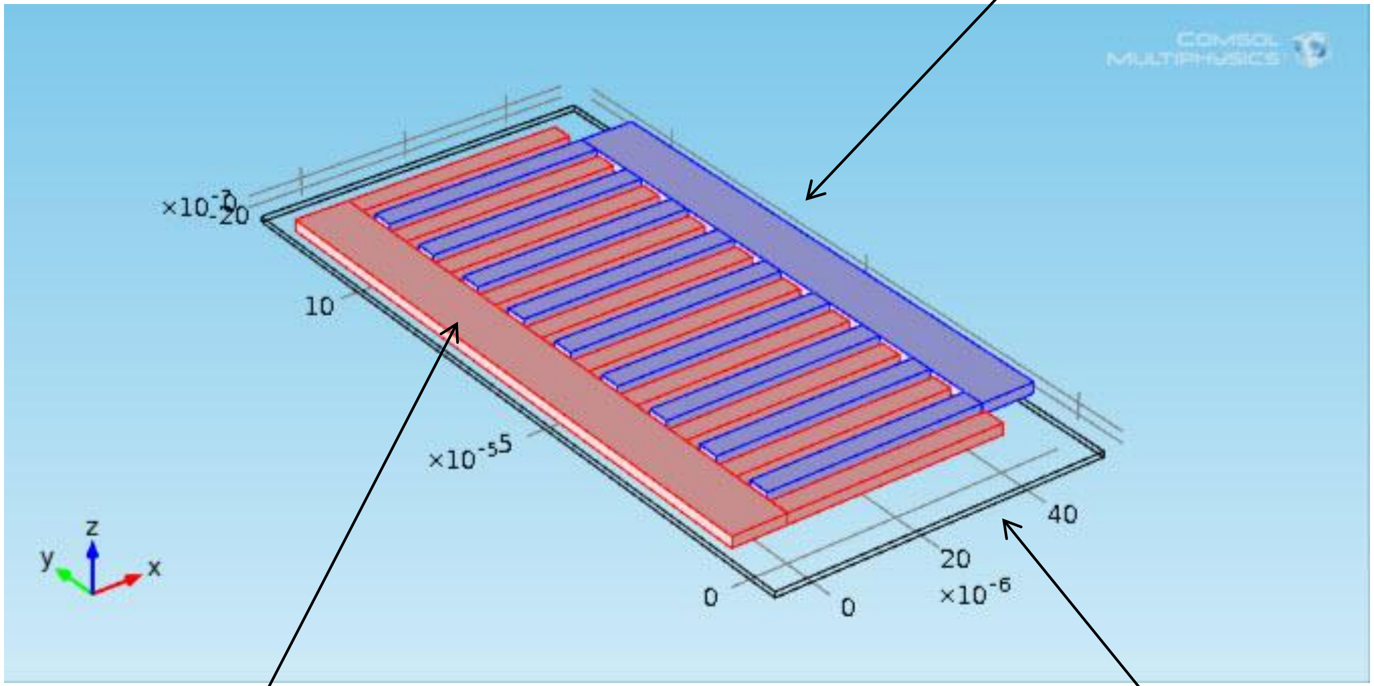
However, only in more recent times has the design of MEMS been approached in a systematic way employing automated optimal design.

Accordingly, the design problem is set up as a problem of non-linear multi-objective optimization of design criteria subject to a set of constraints.

The approach implies suitable computational environments made available by the progress in artificial intelligence, where modelling tools are integrated with soft computing tools .

Comb drive MEMS
3D geometry

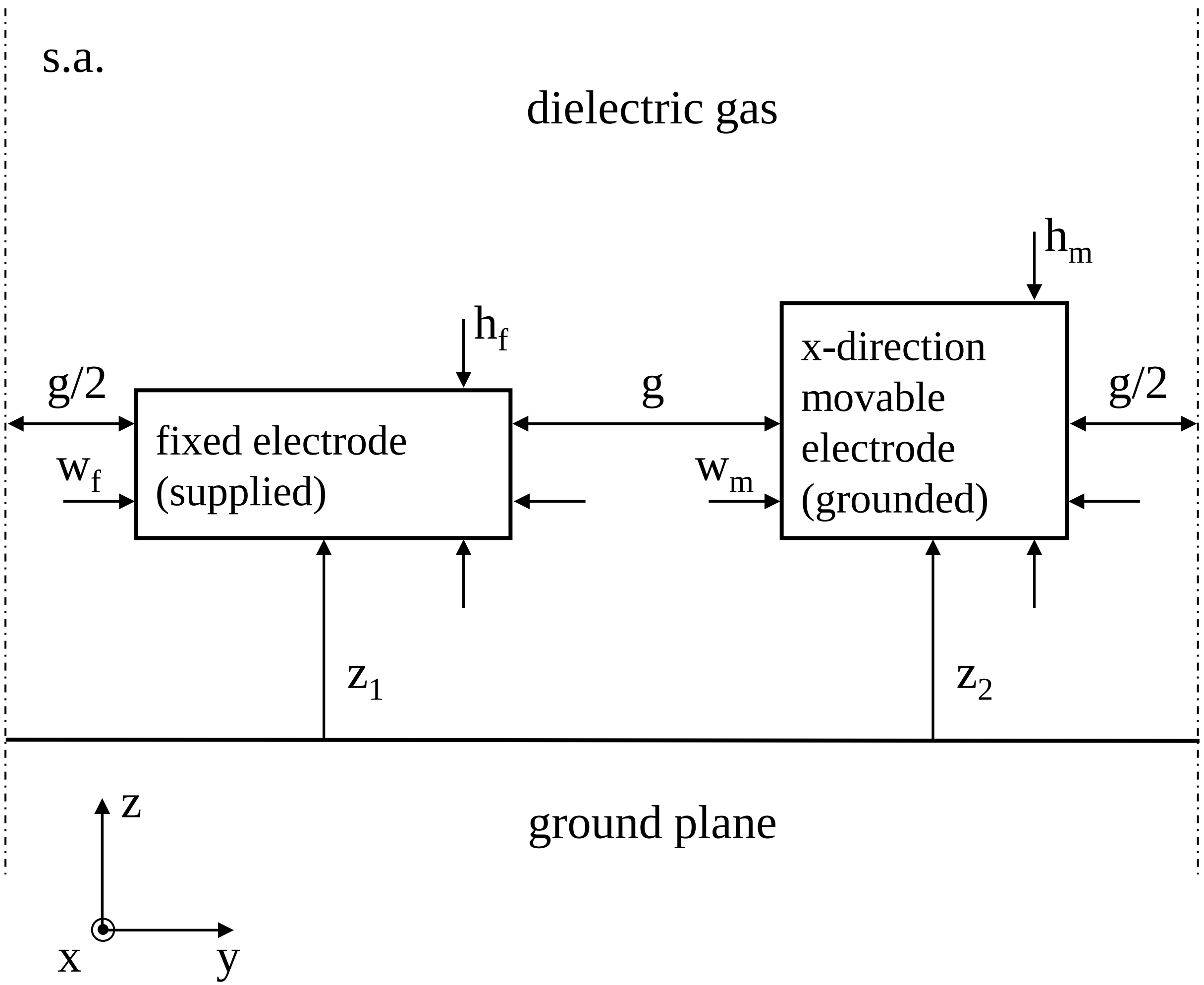
9 movable electrodes (u=0 V)



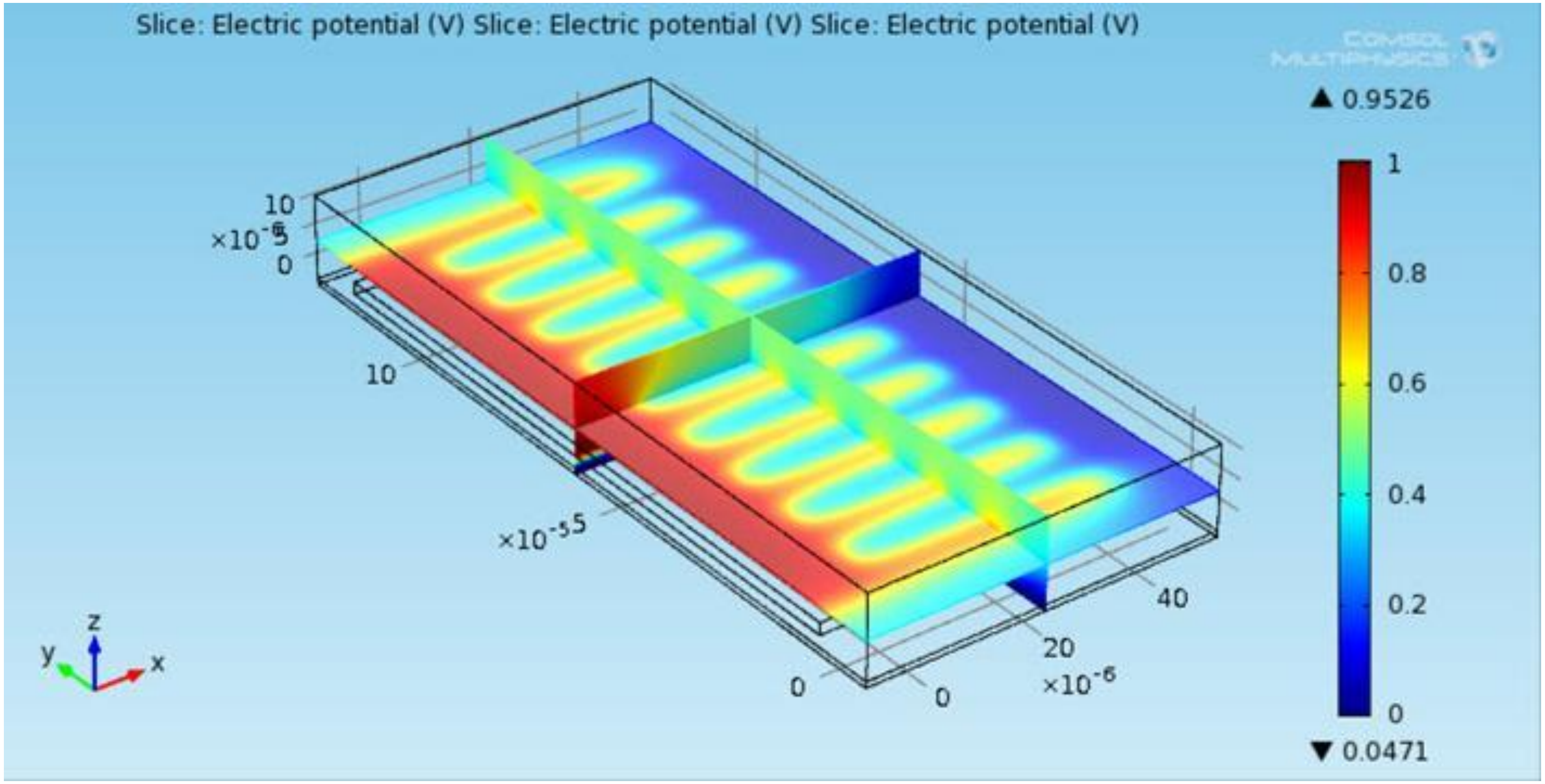
10 fixed electrodes (u=1V)

grounded substrate

Comb drive MEMS: cross-sectional view



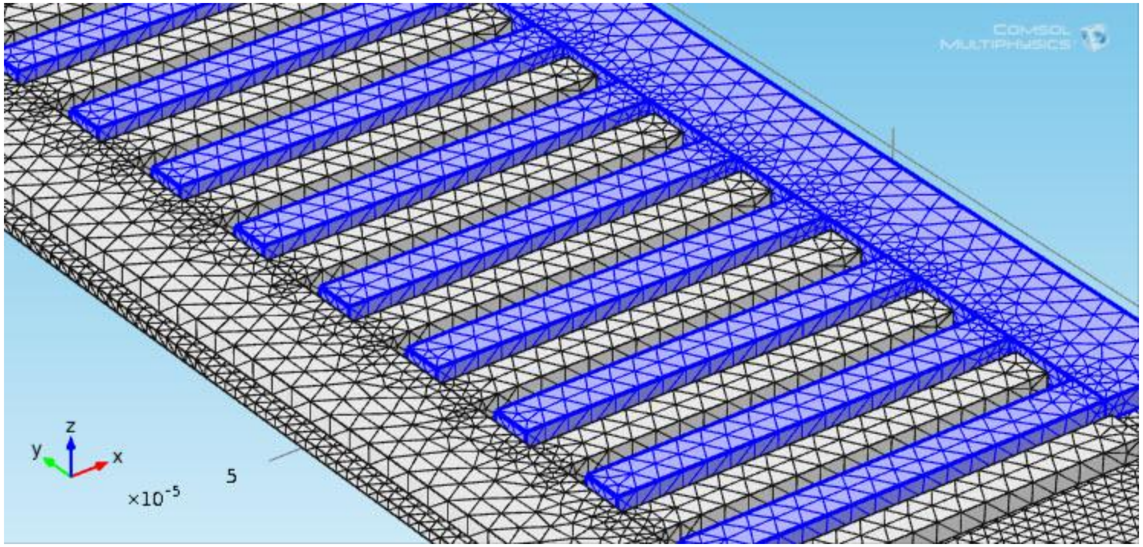
Comb drive MEMS
Field analysis



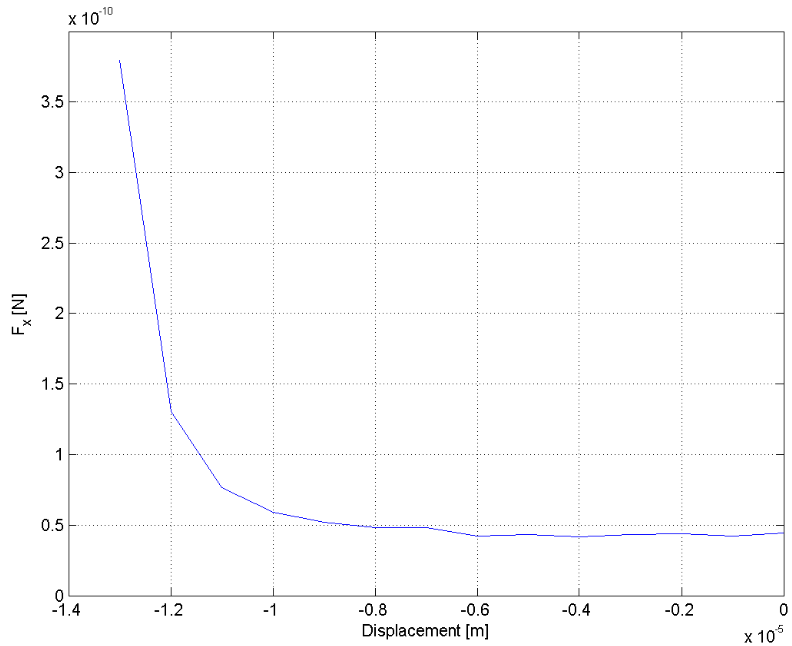
Laplace equation + Maxwell stress tensor

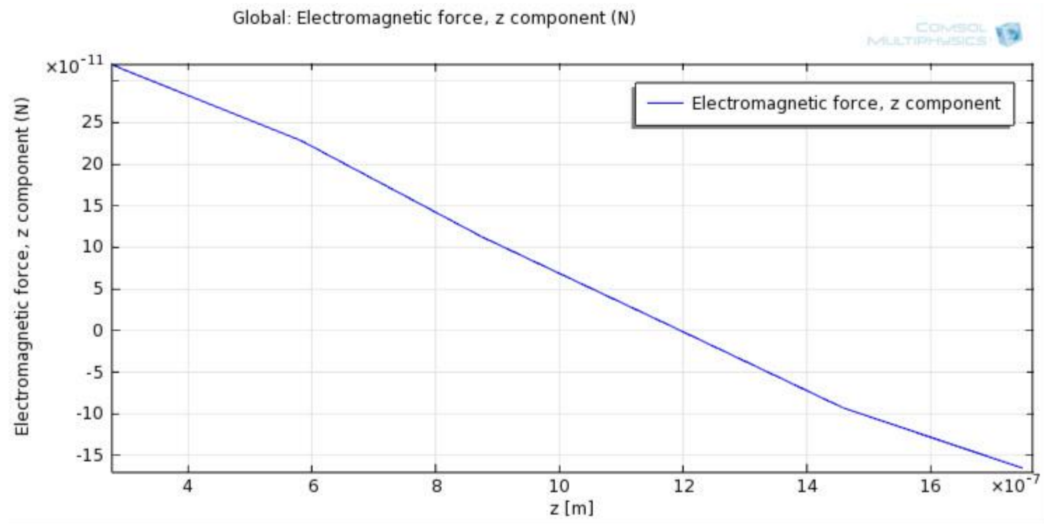
Comb drive MEMS - FE mesh

Mesh characteristic	Value
Minimum element quality	0.2818
Average element quality	0.7927
Tetrahedral elements	170840
Triangular elements	34468
Edge elements	2812
Vertex elements	172
Maximum element size	2.64 μm
Minimum element size	0.0264 μm
Resolution of curvature	0.2
Resolution of narrow regions	1
Maximum element growth rate	1.3



Comb drive MEMS
Force simulation





Levitation force F_z vs z-directed displacement (side effect)

Drive force F_x vs x-directed displacement (main effect)

Approximately, it turns out to be:

$$F_z=k(z-z_0)$$

"electrostatic spring" Constant k

movable electrode equilibrium height z_0

Comb drive MEMS - Optimal design problem

Optimal shape design problem

The goal of the optimal shape design problem is to find the family of geometries which maximise the x-directed drive force between movable and fixed electrodes, and simultaneously mimimise the z-directed levitation force (*electrostatic spring* effect).

Four-dimensional design space

Design variables: width and height of movable and fixed electrodes, respectively.
Design vector $a = (w_m, w_f, h_m, h_f)$.
Range: from 2 to 8 μm.
Discrete-valued (step 0.1 μm).

Comb drive MEMS - Optimal design problem (II)

Two-dimensional objective space

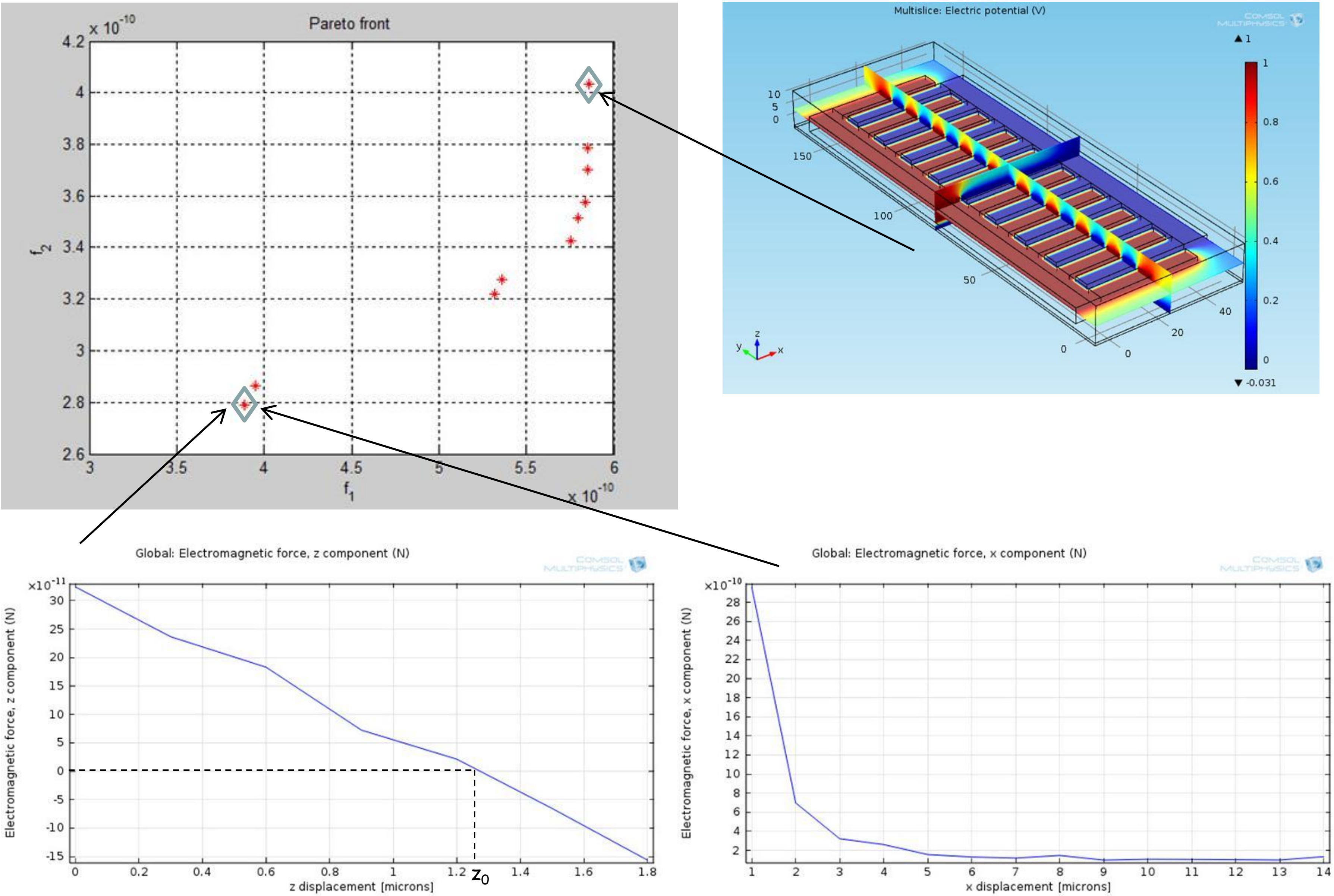
Vector of objective functions $F = (f_1,f_2)$ with

drive $f_1(a) = F_x(x,a)$ for $z = 0$ and $-13 \leq x \leq 0$ μm,
to be maximised with respect to a,

levitation $f_2(a) = F_z(z,a)$ for $x = -13$ μm and $0 \leq z \leq 4$ μm,
to be minimised with respect to a.

Both f_1 and f_2 are subject to the solution of the field analysis problem.

Comb drive MEMS - Optimal design results



Comb drive MEMS - Optimal design results
X and F coordinates of individuals in the final generation

Width of mobile fingers w_m [µm]	Width of fixed fingers w_f [µm]	Height of mobile fingers h_m [µm]	Height of fixed fingers h_f [µm]	F_x drive force [N] $x*10^{-10}$	Slope of F_z vs. z [Nm^{-1}] $x*10^{-10}$
6	6	6.2	6.1	3.8848	2.7915
7.7	7.8	7.7	7.8	5.8568	4.0328
7.1	7.3	7.5	7.4	5.3189	3.2187
6.1	6.1	6.2	6.1	3.9496	2.862
7.6	7.7	7.8	7.9	5.8543	3.7876
7.7	7.7	7.8	7.8	5.8491	3.702
7.6	7.8	7.7	7.8	5.8385	3.5745
7.1	7.2	7.5	7.4	5.3614	3.2781
7.5	7.7	7.7	7.8	5.7529	3.4235
7.7	7.8	7.7	7.8	5.7975	3.5168

CONCLUSION

While there have been significant improvements in the capabilities in the area of MO design, the uptake by industrial designers has been somewhat limited. There are, possibly, two reasons for this.

The first is that the evidence, at the industrial level, that computer-based optimisation processes can actually enhance a designer's ability to create a better product has been lacking.

The second relates to the fact that most optimisation packages currently available only handle a single objective and a limited number of design variables.

In fact, suitable optimisation systems, with no restriction in the size of the design space to be explored, and with simple and flexible expressions of objectives and constraints, would help match the needs of the designer.

P. Di Barba, Multiobjective Shape Design in Electricity and Magnetism, Springer, 2010.