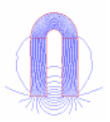
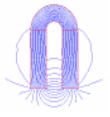
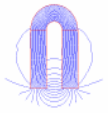


Software Methods and Tools for the Design and Optimization of Electromechanical Devices



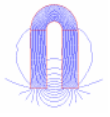
Lecture 2: The Numerical Solution of the Field Equations

Simple Finite Elements



Outline

- The field equations
- Representing surfaces
- One-Dimensional finite elements
- Two-Dimensional finite elements
- A physical interpretation of the matrix



The Field Equations

- Electromagnetic system performance is described by Maxwell's Equations:

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

$$\nabla \times \boldsymbol{E} = \frac{-\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \bullet \boldsymbol{D} = \rho$$

$$\nabla \bullet \boldsymbol{B} = 0$$

- Note that these equations say nothing about relationships between $\boldsymbol{B}$ and $\boldsymbol{H}$ or $\boldsymbol{E}$ and $\boldsymbol{J}$

Materials

- Materials – or rather their physical properties – create the links between the fields..
- The Constitutive Relationships:

$$\boldsymbol{B} = \mu \boldsymbol{H}$$

$$\boldsymbol{J} = \sigma \boldsymbol{E}$$

$$\boldsymbol{D} = \varepsilon \boldsymbol{E}$$

Permeability

Conductivity

Permittivity

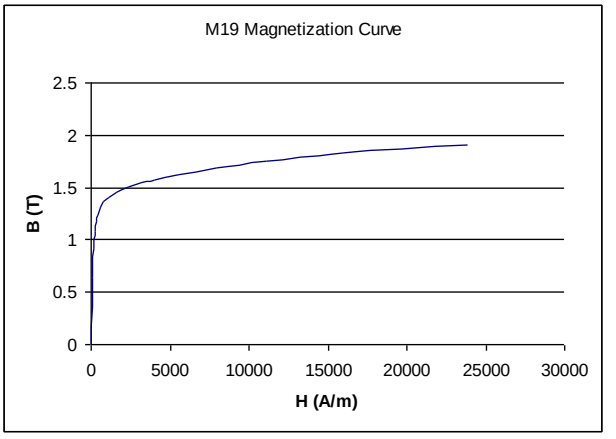
Materials

- Thus a high permeability will create a large local magnetic flux from a magnetic field..
- *Materials are used to control the shape of magnetic and electric fields…*
- Basic constants:

Permeability of free space $\mu_0 = 4\pi \cdot 10^{-7}$ ***Henries / m***

Permittivity of free space $\varepsilon_0 = 8.854 \cdot 10^{-12}$ ***Farads / m***

Material Permeability



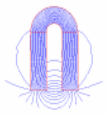
Low Frequency Magnetics (1)

- Consider a situation in which f=0, i.e. statics:

$$\frac{\partial}{\partial t} = 0$$

Maxwell's Equations reduce to:

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}$$
$$\nabla \times \boldsymbol{E} = 0$$
$$\nabla \bullet \boldsymbol{D} = \rho$$
$$\nabla \bullet \boldsymbol{B} = 0$$



Low Frequency Magnetics (2)

- Adding in the constitutive relationship:

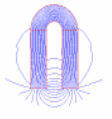
$$\boldsymbol{B} = \mu \boldsymbol{H}$$

$$\nabla \times \boldsymbol{B} = \mu \boldsymbol{J}$$

In two-dimensions, B has two components – the vector lies in the x-y plane. However, since

$$\nabla \bullet \boldsymbol{B} = 0$$

A substitution of the form $\nabla \times \boldsymbol{A} = \boldsymbol{B}$ is allowed



Low Frequency Magnetics (3)

- This results in the "curl-curl" equation

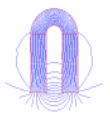
$$\nabla \times \nabla \times A = -\mu J$$

Which reduces to $\nabla^2 A = -\mu J$

This is Poisson's Equation. If $\boldsymbol{J}$=0, it is Laplace

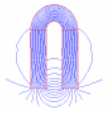
Note that $\boldsymbol{A}$ has only one component: $\boldsymbol{A_z}$, in two dimensions

Half the computing compared to B



The Field as a Surface (1)

- Poisson's equation provides a relationship between:
  - The Field Variable, ***A***
  - The geometric position, x and y
  - The material property, μ
  - The electric current density, ***J***

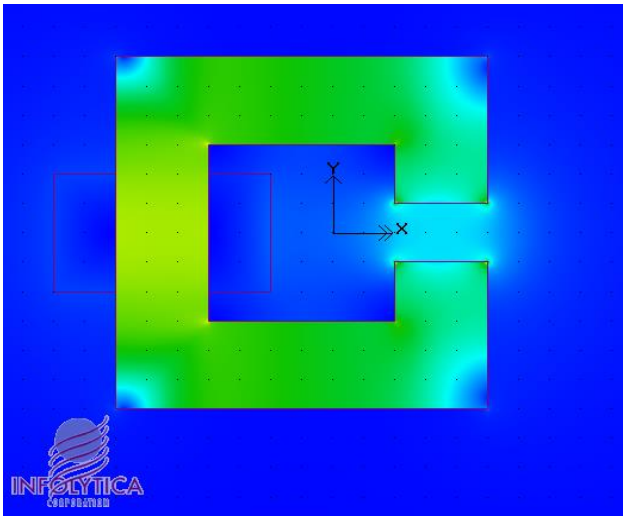


The Field as a Surface (2)

- So – we have:
  - A = A(x,y)
  - μ = μ(x,y)
  - J = J(x,y)
- ***A*** can be considered as a surface over the geometric space…

The Field as a Surface (3)

The different colours represent the height of the function above the plane..



The Field as a Surface (4)

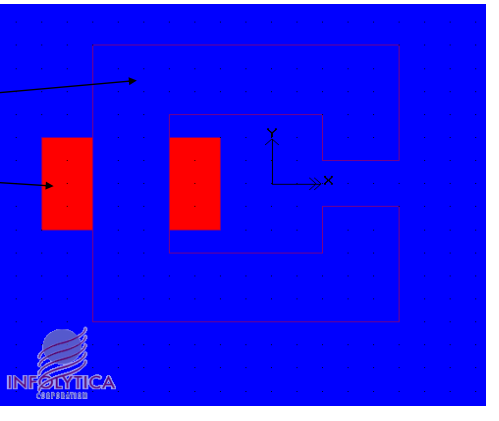
The output plot is a "response" to the input data:

The geometry

The excitations

Governed by Poisson's Equation:

$$\nabla^2 A = -\mu J$$



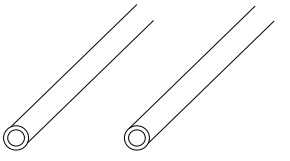
The Field as a Surface (5)

- Note – this is a "physical" rather than a "mathematical" view…
- So the problem is:
  - How do we find the surface?
- Step 1:
  - How can surfaces be modelled?
    - Consider a one-dimensional problem…
      - the "surface" is a curve e.g. the voltage distribution along a buried pipeline..

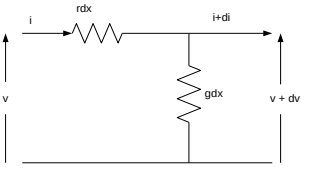
A One-Dimensional Problem (1)

- Consider a buried pipeline being used to carry signals…
  - E.g. a measurement of the integrity of a gas pipeline.
- Problem: find the voltage distribution

(ref: Silvester and Ferrari, Finite Elements for Electrical Engineers)



Buried Pipes



Equivalent Circuit of short length, dx, of pipe.

A One-Dimensional Problem (2)

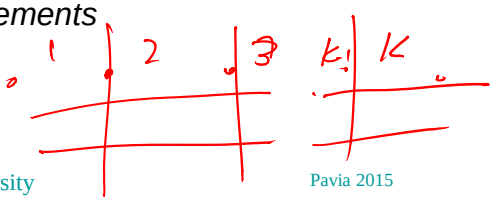
- This problem has an analytical solution.. But we will not use that..
- In the finite element approach, the transmission line equations will not be solved directly.. (true of most f.e. approximations)
- Instead – use the physical principle that *the voltage along the line will adjust itself to minimize the power loss.*

A One-Dimensional Problem (3)

- The steps to solve this problem are:
  - Express the power, W, lost in the line in terms of the voltage distribution, v(x):

$$W = W\{v(x)\}$$

  - Subdivide the entire transmission line into K finite sections = *elements*



A One-Dimensional Problem (4)

Approximate the voltage v(x) using a separate set of approximating expressions in each element:

$$v(x) = \sum_{i=1}^{M} v_i f_i(x)$$

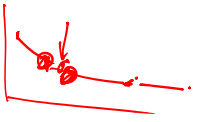
f_i are some convenient known functions (chosen in advance), v_i are unknown coefficients (and there are M of them)

Express the power in each element in terms of $f_i(x)$ and the M undetermined coefficients, v_i

A One-Dimensional Problem (5)

Because f_i are chosen in advance, they are known. The power is thus a function of the unknowns, v_i..

$$W = W(v_1, v_2, \ldots, v_M)$$



Introduce constraints on the *MK* coefficients to ensure the voltage is continuous from element to element.

The ensemble of elements will possess some *N* degrees of freedom, $N \le MK$.

Finally minimize the power by varying each coefficient v_i in turn subject to the constraint that the voltage along the line must vary in a continuous fashion..

A One-Dimensional Problem (6)

$$\frac{\partial W}{\partial v_i} = 0, \quad i = 1,\ldots,N$$

This minimization determines the coefficients and thus produces an approximate expression for the voltage along the line..

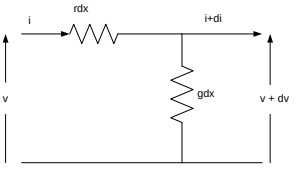
Questions:

How many elements?
What sort of approximating functions?
How are constraints introduced?
What type of minimization technique?

The answers define whole families of finite element methods..

A One-Dimensional Problem (7)

- Explicit expression for power:



Equivalent Circuit of short length, dx, of pipe.

Consider the section, dx.

Power entering is: $W_{in} = vi$

Power leaving is: $W_{out} = (v + dv)(i + di)$

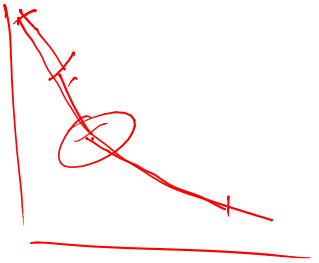
Neglecting second order terms, the power lost in dx is: $dW = vdi + idv$

And $di = g \cdot dx \cdot v, \quad i = \frac{1}{r} \cdot \frac{dv}{dx}$

A One-Dimensional Problem (8)

Thus the power loss per unit length is:

$$\frac{dW}{dx} = -gv^2 - \frac{1}{r}\cdot\left(\frac{dv}{dx}\right)^2$$



And the total loss for the whole line is:

$$W = -\int_0^L \left\{ gv^2 + \frac{1}{r}\cdot\left(\frac{dv}{dx}\right)^2 \right\} dx$$

Now use piecewise-straight approximations for the voltage distribution along the line.. i.e. assume the *voltage varies linearly along each of the K segments (elements)..*

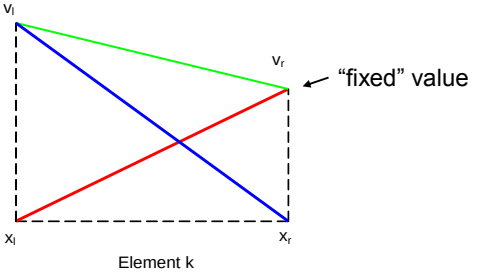
Note that the voltage needs to be continuous for differentiation but the slope does not have to be…

A One-Dimensional Problem (8)

Describe the voltage in a segment by:

$$v = \frac{x_{(k)r} - x}{x_{(k)r} - x_{(k)l}}\cdot v_{(k)l} + \frac{x - x_{(k)l}}{x_{(k)r} - x_{(k)l}}\cdot v_{(k)r}$$

Where (k) is the element, r is the right end and l the left



A One-Dimensional Problem (9)

Express this, for an element, as: $v = \alpha_l(x)v_l + \alpha_r(x)v_r$

where $\alpha_l(x) = \dfrac{x_r - x}{x_r - x_l}$

and $\alpha_r(x) = \dfrac{x - x_l}{x_r - x_l}$

So the total power lost in the line is:

$$W = \sum_{k=1}^{K} W_k$$

where $W_k = -\int_{x_{(k)l}}^{x_{(k)r}} \left\{ gv^2 + \frac{1}{r}\left(\frac{dv}{dx}\right)^2 \right\} dx$

A One-Dimensional Problem (10)

Putting it all together:

$$W_k = -\frac{1}{r_k}\int_{x_{(k)l}}^{x_{(k)r}} \left(v_l \frac{d\alpha_l}{dx} + v_r \frac{d\alpha_r}{dx} \right)^2 dx - g_k \int_{x_{(k)l}}^{x_{(k)r}} \left(v_l\alpha_l + v_r\alpha_r \right)^2 dx$$

Assume that r and g (the resistance and conductance) are constant in an element.

The power lost in an element can be expressed as:

$$W_k = -\begin{bmatrix} v_l & v_r \end{bmatrix} \left[\frac{1}{r_k} S + g_k T \right] \begin{bmatrix} v_l \\ v_r \end{bmatrix}$$

$$S_{ij} = \int_{x_l}^{x_r} \frac{d\alpha_i}{dx}\frac{d\alpha_j}{dx} dx$$

$$T_{ij} = \int_{x_l}^{x_r} \alpha_i \alpha_j dx$$

i and j take on both values, r and l

A One-Dimensional Problem (11)

Constructing the matrices:

$$L_k = x_{(k)r} - x_{(k)l}$$

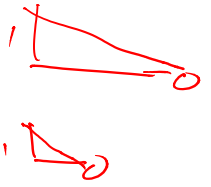
Define a normalized local coordinate:

$$\xi = \frac{x - x_l}{L_k}$$

So

$$\alpha_l(\xi) = 1 - \xi$$

$$\alpha_r = \xi$$



Now

$$v = v_l \alpha_l(\xi) + v_r \alpha_r(\xi)$$

A One-Dimensional Problem (12)

Consider S and T:

$$W_k = -\begin{bmatrix} v_l & v_r \end{bmatrix} \left[\frac{1}{r_k} S + g_k T \right] \begin{bmatrix} v_l \\ v_r \end{bmatrix}$$

$$S_{ij} = \int_{x_l}^{x_r} \frac{d\alpha_i}{dx} \frac{d\alpha_j}{dx} dx$$

$$T_{ij} = \int_{x_l}^{x_r} \alpha_i \alpha_j dx$$

But

Independent of the element length

$$\frac{d\alpha}{dx} = \frac{d\alpha}{d\xi}\frac{d\xi}{dx} = \frac{d\alpha}{d\xi}\frac{1}{L_k}$$

A One-Dimensional Problem (13)

$$W_k = -\begin{bmatrix} \mathrm{v_l} & \mathrm{v_r} \end{bmatrix} \left[\frac{1}{r_k} S + g_k T \right] \begin{bmatrix} v_l \\ v_r \end{bmatrix}$$

$$W_k = -\begin{bmatrix} \mathrm{v_l} & \mathrm{v_r} \end{bmatrix} \left[\frac{1}{r_k} \frac{1}{L_k} \mathbf{S} + g_k L_k \mathbf{T} \right] \begin{bmatrix} v_l \\ v_r \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} \int_0^1 \frac{d\alpha_l}{d\xi} \frac{d\alpha_l}{d\xi} d\xi & \int_0^1 \frac{d\alpha_l}{d\xi} \frac{d\alpha_r}{d\xi} d\xi \\ \int_0^1 \frac{d\alpha_r}{d\xi} \frac{d\alpha_l}{d\xi} d\xi & \int_0^1 \frac{d\alpha_r}{d\xi} \frac{d\alpha_r}{d\xi} d\xi \end{bmatrix}$$

$$\mathbf{T} = \begin{bmatrix} \int_0^1 \alpha_l \alpha_l d\xi & \int_0^1 \alpha_l \alpha_r d\xi \\ \int_0^1 \alpha_r \alpha_l d\xi & \int_0^1 \alpha_r \alpha_r d\xi \end{bmatrix}$$

$$\int_0^1 \alpha_l \alpha_r d\xi = \int_0^1 (1-\xi)\xi d\xi = +\frac{1}{6}$$

$$\int_0^1 \alpha_l \alpha_l d\xi = \int_0^1 \xi^2 d\xi = +\frac{1}{3}$$

$$\int_0^1 \frac{d\alpha_l}{d\xi} \frac{d\alpha_r}{d\xi} d\xi = \int_0^1 (-1)(1) d\xi = -1$$

$$\int_0^1 \left(\frac{d\alpha_l}{d\xi} \right)^2 d\xi = \int_0^1 (1)^2 d\xi = +1$$

A One-Dimensional Problem (14)

So

$$\mathbf{S} = \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix}$$

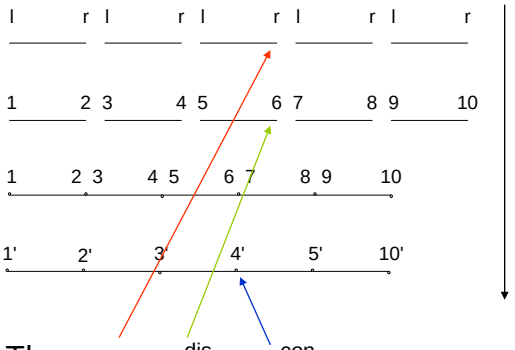
$$\mathbf{T} = \frac{1}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

And the power loss in a single element is:

$$W_k = -\begin{bmatrix} \mathrm{v_l} & \mathrm{v_r} \end{bmatrix} \left(\frac{1}{r_k L_k} \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} + \frac{g_k L_k}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right) \begin{bmatrix} \mathrm{v_l} \\ \mathrm{v_r} \end{bmatrix}$$

A One-Dimensional Problem (15)

Now we need to look at the whole line…



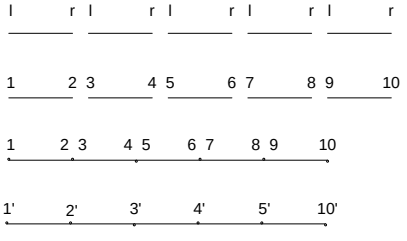
The disjoint segments need to be connected together and then their end voltages set up to be equal where they join to provide continuity..

Thus $v_{3r} = v_6^{dis} = v_{4'}^{con}$

A One-Dimensional Problem (16)

The 10 disconnected voltages are related to the 6 connected ones through:

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \end{bmatrix}_{dis} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} \begin{bmatrix} v_{1'} \\ v_{2'} \\ v_{3'} \\ v_{4'} \\ v_{5'} \\ v_{10'} \end{bmatrix}_{con}$$



$$V_{dis} = CV_{con}$$

A One-Dimensional Problem (17)

Writing the total power as the sum of the element powers:

$$W = -V_{dis}^T \begin{bmatrix} \frac{\mathbf{S}}{r_1 L_1} + g_1 L_1 \mathbf{T} & & & & \\ & \frac{\mathbf{S}}{r_2 L_2} + g_2 L_2 \mathbf{T} & & & \\ & & \ddots & & \\ & & & & \frac{\mathbf{S}}{r_5 L_5} + g_5 L_5 \mathbf{T} \end{bmatrix} V_{dis} = -V_{dis}^T M_{dis} V_{dis}$$

and

$$W = -V_{con}^T C^T M_{dis} C V_{con}$$

$$M = C^T M_{dis} C$$

A One-Dimensional Problem (18)

If the resistance and conductance per unit length are the same for all elements and all elements are of length L_e, the final connected matrices for the 5 element model are:

$$\frac{1}{r} C^T \mathbf{S} C = \frac{1}{L_e r} \begin{bmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & 2 & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix}$$

A One-Dimensional Problem (19)

And, for the T matrix:

$$gC^T\mathbf{T}C = \frac{L_e g}{6}\begin{bmatrix} 2 & 1 & & & & \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & \\ & & 1 & 4 & 1 & \\ & & & 1 & 4 & 1 \\ & & & & 1 & 2 \end{bmatrix}$$

A One-Dimensional Problem (20)

The total power loss is:

$$W = -V_{con}^T\left(\frac{1}{r}C^T\mathbf{S}C + gC^T\mathbf{T}C\right)V_{con}$$

$$W = -V_{con}^T M V_{con}$$

The last step is to minimize the power loss. All the voltages are free to vary except one – the one at the source end of the pipe…

$$\frac{\partial W}{\partial v_k} = 0, \quad k = 1,2,\ldots,N-1$$

A One-Dimensional Problem (21)

Differentiating gives:

$$\begin{bmatrix} M_{1'1'} & M_{1'2'} & \cdots & & & \\ M_{2'1'} & M_{2'2'} & & & & \\ \cdot & & \cdot & & & \\ \cdot & & \cdot & & & \\ & & & M_{5'5'} & M_{5'10'} & \\ & & & M_{10'5'} & M_{10'10'} & \end{bmatrix} \begin{bmatrix} v_{1'} \\ v_{2'} \\ \cdot \\ \cdot \\ v_{5'} \\ v_{10'} \end{bmatrix} = 0$$

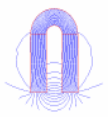
However, $v_{10'}$ is actually fixed – it is not an unknown

A One-Dimensional Problem (22)

Moving the known values to the right hand side gives:

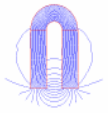
$$\begin{bmatrix} M_{1'1'} & M_{1'2'} & \cdots & \\ M_{2'1'} & M_{2'2'} & & \\ \cdot & & \cdot & \\ \cdot & & & \cdot \\ & & & M_{5'5'} \end{bmatrix} \begin{bmatrix} v_{1'} \\ v_{2'} \\ \cdot \\ \cdot \\ v_{5'} \end{bmatrix} = \begin{bmatrix} -M_{1'10'}v_{10'} \\ -M_{2'10'}v_{10'} \\ \cdot \\ \cdot \\ -M_{5'10'}v_{10'} \end{bmatrix}$$

This equation represents exactly as many simultaneous algebraic equations as there are free nodes…



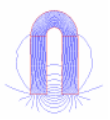
Approximation Errors

- What determines the error in modelling the solution by finite elements?
  - It depends how many elements are used…
  - It depends on the shape of the curve being modelled…
  - It depends on the functions used in an element…



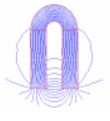
Summarize

- Divide the problem into small pieces (elements).
- Assume the field behavior over the element can be modeled using a linear combination of simple functions
- Specify the functions
- Determine the coefficients of each function by minimizing an expression for a physical quantity (e.g. energy)



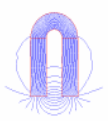
Summarize

- Join all the elements together to cover the complete problem domain.
- Connect equivalent points
- Minimize the global quantity
- Solve the resulting set of linear equations to determine the potential values everywhere.



Two-Dimensional Problems

- Not too many real magnetic devices can be modeled in one-dimension…
- Two-dimensions, i.e. x and y or r and θ, is the minimum for many devices.
- So, how can the method be extended to 2-D?



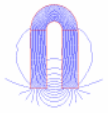
Two-Dimensional Finite Elements (1)

- First.. What element shape should be used for two-dimensional problems?
- In theory, any polygonal shape could be used…
- If we extend the 1-D approach, then use a linear *shape function*, i.e. linear in x and y..

$$U = ax + by + c$$

This is a *first order shape function*..

Note – U is an approximate solution – A is the true solution A=U if the model is accurate



Two-Dimensional Finite Elements (2)

The function has 3 unknowns:

a,b and *c*

So, we need to be able to define 3 equations where U is defined at three points in space (x_1,y_1), (x_2,y_2), (x_3,y_3) to compute them..

So the basic shape has to have three "vertices" or nodes, i.e. a *triangle*.

This is a *simplex* element

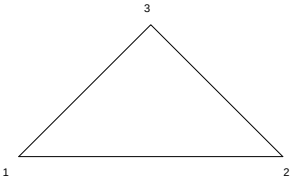
Two-Dimensional Finite Elements (3)

- A simplex is the simplest geometric structure in any particular dimension..
  - In 1-D the simplex is a line
  - In 2-D the simplex is a triangle
  - In 3-D the simple is a tetrahedron
  - …

Two-Dimensional Finite Elements (4)

Three vertices - *nodes*

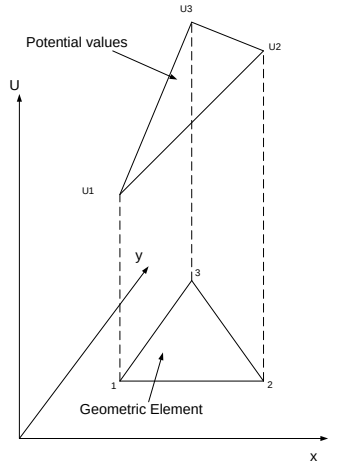
Usually numbered in an *anti-clockwise* direction.



Linear shape function:

$$U = ax + by + c$$

Two-Dimensional Finite Elements (5)



$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

Two-Dimensional Finite Elements (6)

Rewriting:

$$U = \sum_{i=1}^{3} U_i \alpha_i(x, y)$$

$$\alpha_1 = \frac{1}{2A}\{(x_2 y_3 - x_3 y_2) + (y_2 - y_3)x + (x_3 - x_2)y\}$$

α_i is known as a *position* function
Obtained by a cyclic rotation of the subscripts…
They are *interpolatory* functions, i.e.

$$\alpha_i(x_j, y_j) = 0 \quad i \neq j$$

$$\alpha_i(x_j, y_j) = 1 \quad i = j$$

Two-Dimensional Finite Elements (7)

The energy in an element can now be defined:

$$\nabla U = \sum_{i=1}^{3} U_i \nabla \alpha_i$$

$$W^{(e)} = \frac{1}{2} \int \nabla U \bullet \nabla U dS$$

Giving:

$$W^{(e)} = \sum_{i=1}^{3} \sum_{j=1}^{3} U_i \int \nabla \alpha_i \bullet \nabla \alpha_j dS U_j$$

Two-Dimensional Finite Elements (8)

Writing

$$S_{ij}^{(e)} = \int \nabla \alpha_i \bullet \nabla \alpha_j dS$$

The energy in the element can be expressed by:

$$W^{(e)} = \frac{1}{2} U^T S^{(e)} U$$

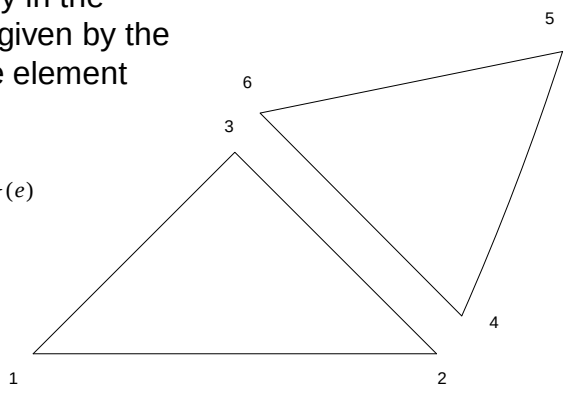
One term in the S matrix can be evaluated (the others are obtained by cyclical permutation of indices)

$$S_{12}^{(e)} = \frac{1}{4A} \{(y_2 - y_3)(y_3 - y_1) + (x_3 - x_2)(x_1 - x_3)\}$$

Two-Dimensional Finite Elements (9)

The energy in the system is given by the sum of the element energies:

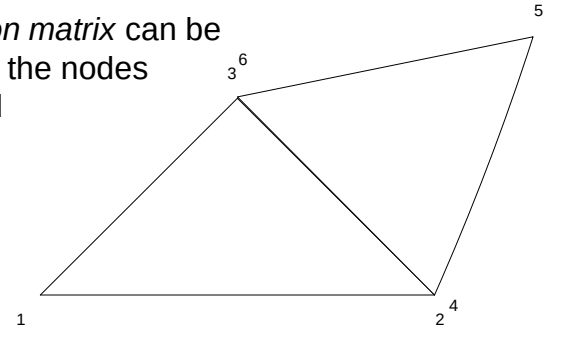
$$W = \sum_e W^{(e)}$$



Two-Dimensional Finite Elements (10)

As in 1-D, the elements must be joined to impose continuity…

A *Connection matrix* can be created and the nodes renumbered



Two-Dimensional Finite Elements (11)

For the two elements, the combined S matrix is:

$$S = \begin{bmatrix} S_{11}^1 & S_{12}^1 & S_{13}^1 & 0 \\ S_{21}^1 & S_{22}^1 + S_{44}^2 & S_{23}^1 + S_{46}^2 & S_{45}^2 \\ S_{31}^1 & S_{32}^1 + S_{64}^2 & S_{33}^1 + S_{66}^2 & S_{65}^2 \\ 0 & S_{54}^2 & S_{56}^2 & S_{55}^2 \end{bmatrix}$$

As before, the solution of the field problem requires the minimization of stored energy

$$\frac{\partial W}{\partial U_k} = 0$$

Two-Dimensional Finite Elements (12)

The solution to this problem is trivial, i.e. U=0, if there are no prescribed boundary values…

If some boundary potentials are fixed, then the *U* vector can be divided into U_f (free) and U_p (prescribed). If the free nodes are numbered first:

$$\frac{\partial W}{\partial U_k} = \frac{\partial}{[U_f]_k}\left[U_f^T U_p^T\right]\begin{bmatrix} S_{ff} & S_{fp} \\ S_{pf} & S_{pp} \end{bmatrix}\begin{bmatrix} U_f \\ U_p \end{bmatrix} = 0$$

Two-Dimensional Finite Elements (13)

This results in:

$$\begin{bmatrix} S_{ff} & S_{fp} \end{bmatrix} \begin{bmatrix} U_f \\ U_p \end{bmatrix} = 0$$

$$S_{ff} U_f = -S_{fp} U_p$$

Solving for U_f results in a solution to the problem

i.e. U is approximately equal to A

Error Estimation

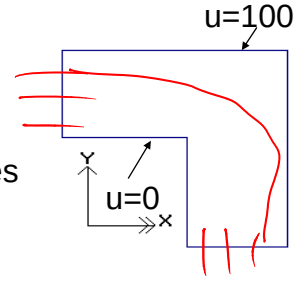
Consider a two-dimensional problem governed by Laplace's equation:

$$\nabla^2 u = 0$$

It is driven by the boundary values:

Either $u=u_{spec}$ on conductor surfaces

Or $\frac{\partial u}{\partial n} = 0$ on symmetry planes.



The stored energy is: $W(u) = \frac{1}{2}\int \nabla u \bullet \nabla u dS$

Error Estimation

Assume that u(x,y) is the true solution to the problem while h(x,y) is a sufficiently differentiable function with a value of 0 at every boundary point where u has a specified value from the boundary conditions..

So (u+θh) – where θ is a scalar parameter – is an approximate solution which has the same prescribed boundary conditions as u.

The energy of this approximate solution is, then:

$$W(u+\theta h) = W(u) + \theta \int \nabla u . \nabla h dS + 0.5\theta^2 \int \nabla h . \nabla h dS$$

Two-Dimensional Finite Elements (14)

- How good is the solution? or Where are the errors?
  - This is a piecewise linear approximation..
  - The best fit for the number of elements
- The solution is EXACT at the prescribed boundary values (the U_p).

$$W(u+\theta h) = W(u) + \theta^2 W(h) - \theta \int h \nabla^2 u dS + \theta \oint h \frac{\partial u}{\partial n} ds$$

Two-Dimensional Finite Elements (15)

$$W(u+\theta h) = W(u) + \theta^2 W(h) - \theta \int h \nabla^2 u dS + \theta \oint h \frac{\partial u}{\partial n} ds$$

Error within the area

Error on the boundary

On the boundary *h* is zero wherever *u* is prescribed. System will try to make *∂u/∂n* zero in some average sense – it will not be zero at each point…

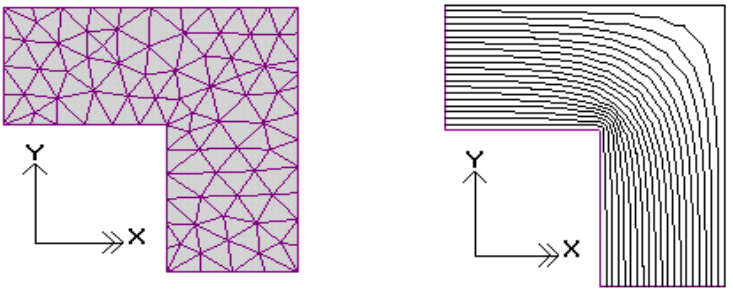
Two-Dimensional Finite Elements (16)

Within the area, the connection between elements ensures that the variation of *u* along the common edge is the same in both elements.

The variation of u normal to the common edge is different on either side of the element – i.e. this variation is discontinuous (we did not impose C^1 continuity on the connection)

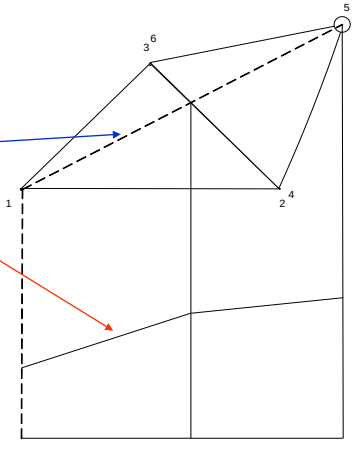
This discontinuity can be considered an indication of the local error.

Two-Dimensional Finite Elements (17)



Two-Dimensional Finite Elements (18)

Value of u along line through elements



Distance along line

Two-Dimensional Finite Elements (19)

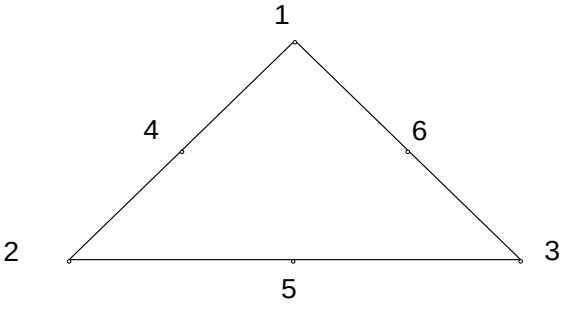
- How can the accuracy be improved?
  - More elements… the more elements, the better the surface shape can be matched..
  - Change the shape functions..
  - Use a higher order, e.g. a quadratic:

$$U = ax^2 + by^2 + cxy + dx + ey + f$$

Now we need 6 points to find the unknowns

Two-Dimensional Finite Elements (20)

The mathematical development follows the same process as before.



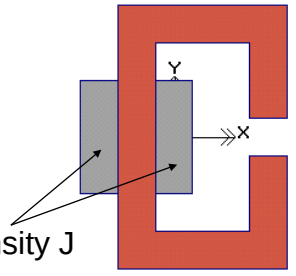
Shape functions can be first, second, third, fourth,… order

Systems can be built to include C^1 continuity..

Two-Dimensional Finite Elements (21)

- What about Poisson's equation?
  - Electromagnets
    - These systems have sources which are not on the boundaries..

$$\nabla^2 A = -\mu_0 J$$



Coil with current density J

Two-Dimensional Finite Elements (22)

The energy related functional is:

$$F(u) = \frac{1}{2}\int \nabla u \bullet \nabla u dS - \mu_0 \int u J dS$$

It can be shown that this will reach a true solution when minimized, the same approach as was taken with Laplace can be used…

The minimum of the functional occurs when u=A, the solution of Poisson's equation.

Two-Dimensional Finite Elements (23)

The finite element formulation proceeds as before. The first term in the energy is just the Laplace term… and can be represented with the same *S* matrix.

Over an element, approximate the current density, J, in the same way as the potential:

$$J = \sum_{i=1}^{3} J_i \alpha_i(x, y)$$

Two-Dimensional Finite Elements (24)

The source term integral is then:

$$\int AJdS = \sum_{i=1}^{3}\sum_{j=1}^{3} A_i \int \alpha_i \alpha_j dS \, J_j$$

Let *T* be defined as:

$$T_{ij}^{e} = \int \alpha_i \alpha_j dS$$

The element "input" energy is then

$$\int AJdS = A^T T^{(e)} J$$

Two-Dimensional Finite Elements (25)

Allowing for the connection of the elements, etc., the discretized equation for the total energy is:

$$F(A) = \frac{1}{2} A^T S A - \mu_0 A^T T J$$

And minimizing:

$$\frac{\partial F}{\partial A_k} = 0$$

Results in the final system:

$$S_{ff} A_f = \mu_0 T J - S_{fp} A_p$$

Two-Dimensional Finite Elements (26)

- Inclusion of material properties
  - So far the development has assumed that the materials are all air…
  - It also assumes that the permeability is fixed
  - In a real device, the permeability, μ, varies within the geometry and is really a function of position:

$$\mu = \mu(x, y)$$

In this case, the permeability has to remain inside the curl operator…

Two-Dimensional Finite Elements (27)

Thus the curl-curl equation is:

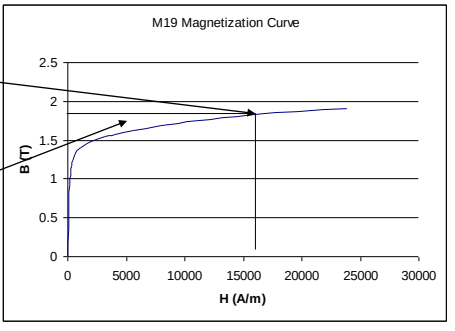
$$\nabla \times \frac{1}{\mu} \nabla \times A = -J$$

In this case, the stored energy now has to take into account the energy in the material – represented by integrating up the B-H curve..

Two-Dimensional Finite Elements (28)

Working point

Energy



Two-Dimensional Finite Elements (29)

The energy in the material is given by:

$$E = \int_0^B H.db$$

The finite element formulation can now be reworked substituting for B and H with A and following the same process as before.

Note that since the permeability is a function of A now, this is a non-linear problem.

Two-Dimensional Finite Elements (30)

A typical term in the S matrix is now given by:

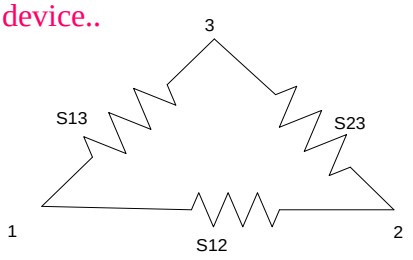
$$S_{12}^{(e)} = \frac{1}{4A\mu}\{(y_2 - y_3)(y_3 - y_1) + (x_3 - x_2)(x_1 - x_3)\}$$

This is a linearised version.. To solve the problem, it requires an iterative process where the μ values for each element are updated as a result of each solution of the linear equations.

It is usual to assume that μ is constant within an element.

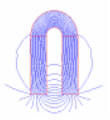
Interpreting the S matrix

- The terms in the S matrix have the dimensions of reluctance..
  - In fact, this could be considered to describe an equivalent circuit representing the magnetic device..



Solving the Equations

- The resultant finite element based set of equations for a magnetostatic problem is VERY large and non-linear…
- A typical problem for an electrical machine may need 10^4 or 10^5 nodes..
- Two solver processes are needed:
  - A LINEAR equation solver
  - A NON-LINEAR process..



Solving the Equations

- Any linear equation solver will work..
- However the system is:
  - Very sparse
  - Can be badly conditioned
- A direct (gaussian elimination) solver has approximately $O(N^3)$ complexity
  - N is the number of degrees of freedom = free nodes.