Magnetic Field Synthesis in an Open-Boundary Region Exploiting Thévenin Equivalent Conditions

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Outline of the presentation

Optimal shape design with Thévenin like conditions

Conditions of equivalence

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Optimal design procedure

The H-shaped electromagnet as a benchmark

Controlling the FE mesh

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Field analysis

Field synthesis

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INTRODUCTION

When solving analysis problems in electricity and magnetism by means of finite-element (FE), often a small part of the field domain includes the region of main interest.

Nonetheless, the analysis of the whole domain has to be performed, even in subdomains of little or no interest. In the case of repeated field analyses, like *e.g.* in design problems, the computational burden grows up.

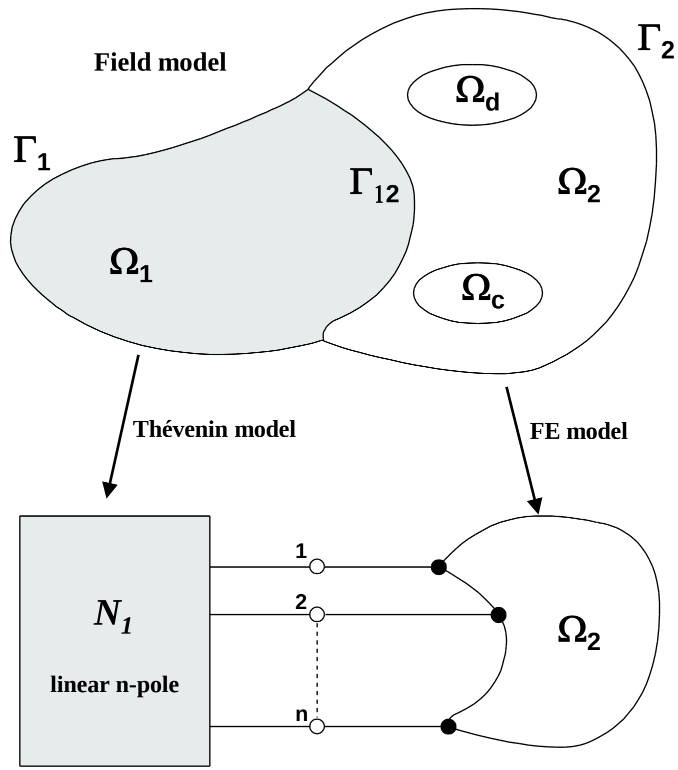
In (Santini and Silvester, 1996) a principle of field diakoptics was presented: the region which is not of interest is replaced by means of the generalized Thévenin theorem.

The systematic use of Thévenin-like conditions in field analysis is now proposed as a useful method to solve optimal shape design problems.

OPTIMAL SHAPE DESIGN WITH THEVENIN-LIKE CONDITIONS

Given a field region Ω_3 , let the controlled region Ω_c , and the region Ω_d in which the design variables x are defined, belong to a subdomain Ω_2 included in Ω_3 . Moreover, let the complementary subdomain $\Omega_1 = \Omega_3 \setminus \Omega_2$ be filled in by linear materials.

Then, an optimal shape design problem defined in Ω_3 can be solved acting only on the finite-element grid discretizing Ω_2 , after replacing Ω_1 with the corresponding Thévenin n-pole N_1 .



CONDITIONS OF EQUIVALENCE

After a cut along the common boundary Γ_{12} , the magnetic effect of Ω_1 on Ω_2 (and *vice versa*) can be restored by means of a line current source J_{12} acting along Γ_{12} *i.e.*

$$-\overline{\nabla}\cdot\left(\mu_2^{-1}\overline{\nabla}u_2\right)=J_2+J_{12}\ \ in\ \ \Omega_2 \quad \text{and} \quad -\overline{\nabla}\cdot\left(\mu_1^{-1}\overline{\nabla}u_1\right)=J_1-J_{12}\ \ in\ \ \Omega_1$$

$$\Rightarrow \quad K_2u_2=J_2+C_2J_{12}\ \ in\ \ \Omega_2 \quad \text{and} \quad K_1u_1=J_1-C_1J_{12}\ \ in\ \ \Omega_1$$

with u magnetic potential.
In particular, an equivalent domain Ω_e replacing Ω_1 can be identified which has the same line current J_{12} as Ω_1

⇓

When J_{12} is the same in both domains Ω_1 and Ω_e , the potential at the common boundary Γ_{12} is the same in $\Omega_1 U \Omega_2$ and in $\Omega_e U \Omega_2$

FINITE-ELEMENT EQUATIONS OF THE ANALYSIS PROBLEM

FE model

n_{p_2} equations for domain Ω_2

$n_T << n_{p_2}$ eq.s for Thévenin-like conditions

n_T continuity equations along Γ_{12}

$$\begin{bmatrix} K_2 & 0 & -C_2 \\ 0 & K_e & I_{n_T} \\ -C_2^t & I_{n_T} & 0 \end{bmatrix} \begin{bmatrix} u_2 \\ u_e \\ J_{12} \end{bmatrix} = \begin{bmatrix} J_2 \\ J_e \\ 0 \end{bmatrix}$$

Stiffness matrix and current density vector for equivalent domain Ω_e

No matrix should be explicitly inverted

$$K_e = \left[C_1^t K_1^{-1} C_1\right]^{-1} \qquad J_e = K_e C_1^t K_1^{-1} J_1$$

Augmented system

n_{p_2} governing equations

$$\left[K_2 + C_2 K_e C_2^t\right] u_2 = J_2 + C_2 J_e$$

modified stiffness matrix → $\left[K_2 + C_2 K_e C_2^t\right]$

modified current density vector → $J_2 + C_2 J_e$

K_e is a dense matrix

THEVENIN EQUIVALENCE FOR THE ANALYSIS PROBLEM

Stiffness matrix and current density vector
for the equivalent domain Ω_e replacing domain Ω_1

$$K_e = \left[C_1^t K_1^{-1} C_1\right]^{-1} \qquad J_e = K_e C_1^t K_1^{-1} J_1$$

It is not necessary to invert matrix K_1 explicitly.
In fact, it turns out to be

$$K_1^{-1} J_1 = u_1\big|_{J_{12}=0} \quad \Rightarrow \quad J_e = K_e C_1^t u_1\big|_{J_{12}=0}$$

$$K_1 u_1 = C_1 J_{12}\big|_{J_1=0, J_{12}=-1} \quad \Rightarrow \quad C_1^t K_1^{-1} C_1 = C_1^t U\left(u_1\big|_{J_1=0, J_{12}=-1}\right)$$

Ω_1 disconnected

Ω_1 inner source free
and unit outer source

The only matrix to be explicitly inverted is $C_1^t U\left(u_1\big|_{J_1=0, J_{12}=-1}\right)$
of size (n_T, n_T) but $n_T \ll n_{p2}$

NON-LINEAR CASE

If Ω_2 incorporates a non-linear magnetic material, stiffness matrix K_2 depends on the unknown potential u_2 .

Formally, the solving system modifies as follows:

$$\left[K_2(u_2)+C_2K_eC_2^t\right]u_2 = J_2 + C_2J_e$$

Given an initial estimate of u_2 , an iterative procedure can be started to solve the system of non-linear equations (e.g. Newton-Raphson algorithm).

OPTIMAL DESIGN PROCEDURE

Derivative-free
optimisation algorithm

Minimum of the
objective function

Core equation
for field analysis

$$\left[K_2 + C_2 K_e C_2^t\right] u_2 = J_2 + C_2 J_e$$

Magnetic field in the
reduced domain Ω_2

Arrays K_e and J_e are computed
only at the start

K_2 , C_2 , and J_2 are updated
according to the variations
determined in Ω_2 by the
optimisation algorithm

OPTIMAL DESIGN PROCEDURE (II)

Tentative cost of a derivative-free opt procedure

on the whole field domain Ω_3

$$c_3 \approx k(n_v, n_c)\, n_f\, n_i(\varepsilon)\, n_{p_3}^2$$

variables, constraints, objectives, convergence it.s, mesh nodes

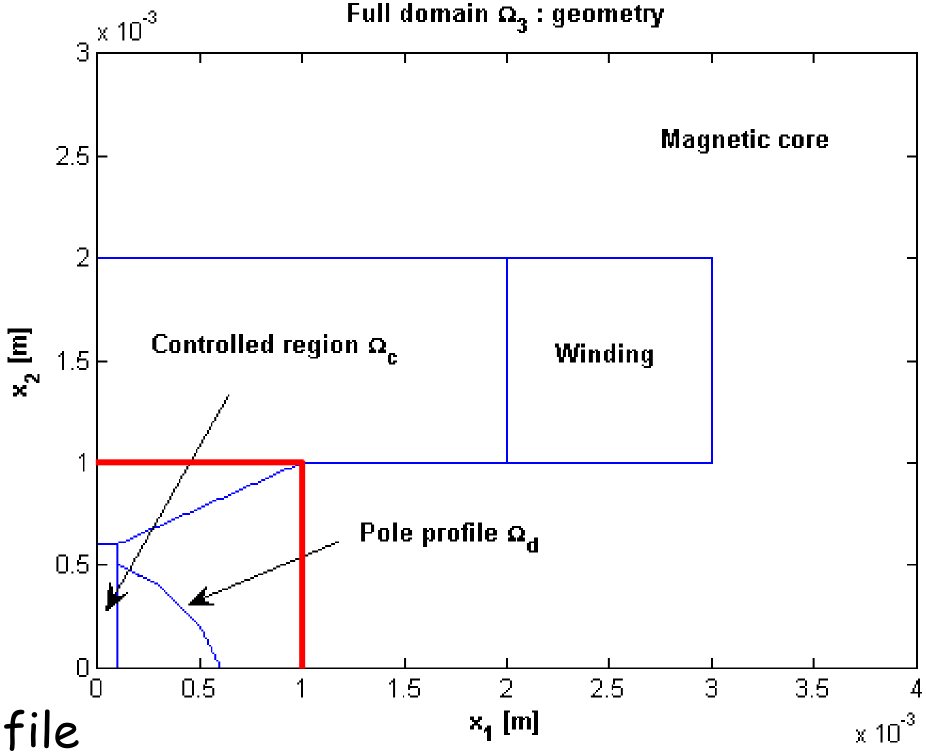
on the field subdomain Ω_2

$$c_2 \approx k(n_v, n_c)\, n_f\, n_i(\varepsilon)\, n_{p_2}^2$$

$$\Rightarrow \quad \frac{c_2}{c_3} = \left(\frac{n_{p_2}}{n_{p_3}}\right)^2 < 1$$

possible cost reduction

THE H-SHAPED ELECTROMAGNET AS A BENCHMARK



The device

Prescribed induction field at the air gap

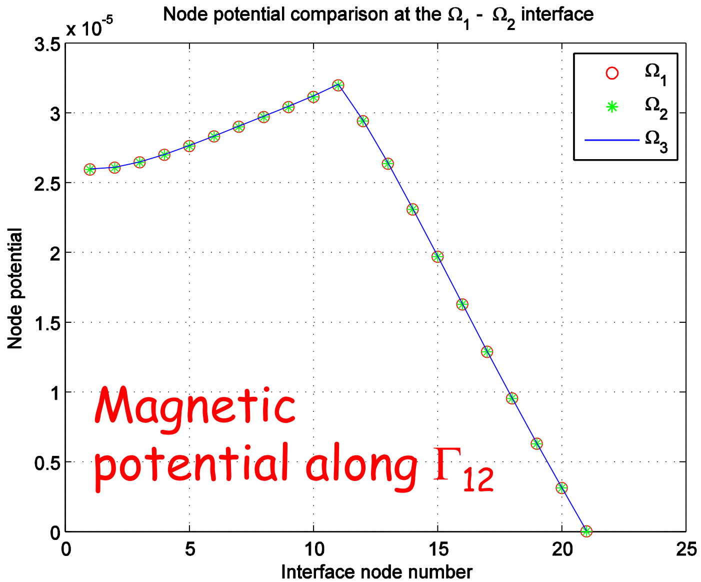
$$\overline{B} = (B_0, 0)$$

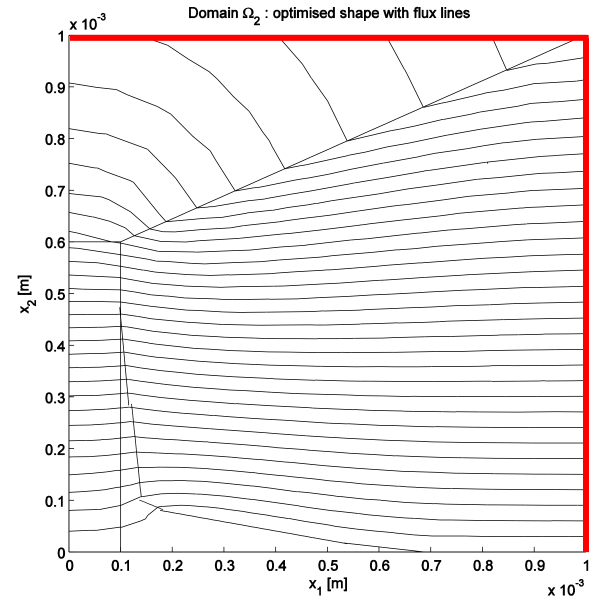
Design variables:
rectangular coordinates of a set of moving nodes discretizing the pole profile

The objective function, to be minimised

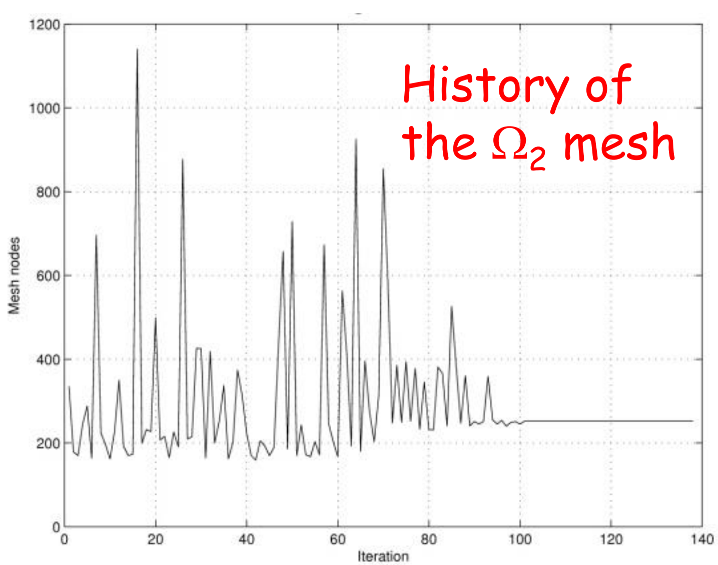
$$f(\xi) = \sup_{\Omega_c} |B_1(x,\xi) - B_0| + \sup_{\Omega_c} |B_2(x,\xi)| \quad , \quad x \in \Omega_c \quad , \quad \xi \in \Omega_d$$

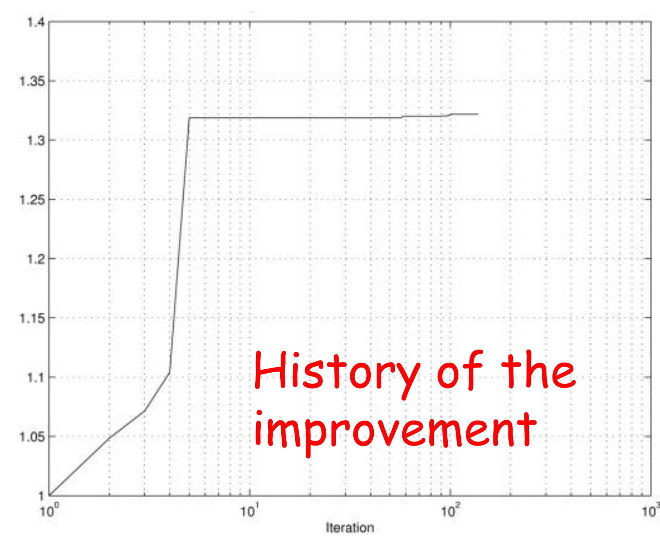
H-SHAPED ELECTROMAGNET - RESULTS





Improved pole shape and flux lines





FE MESH CONTROL

The number of nodes along the Thévenin boundary should be kept constant, and their position should be prescribed for the equivalence to be exact.

Using a commercial code of FEA the constraint of prescribed node distribution along the Thévenin boundary should be relaxed; moreover, the construction of appropriate connectivity matrices, adjusting the nodes of subdomain Ω_2 is necessary.

When Ω_1 and Ω_2 are connected, let the common boundary Γ_{12} exhibit n_T nodes. In turn, when Ω_1 and Ω_2 are not connected, let n_{T1} (n_{T2}) be the number of nodes along the corresponding boundary of Ω_1 (Ω_2).

In this case, a line Γ_{12} exhibiting n_T nodes can still be defined as a hinge layer between Ω_1 and Ω_2 (node mismatch) .

FE MESH CONTROL (II)

The system of equations governing the FE model must be recast in matrix form as follows:

n_{p2} eq.s in domain Ω_2

n_{T1} eq.s for Thévenin conditions

n_T continuity eq.s along Γ_{12}

$$\begin{bmatrix} K_2 & 0 & -C_2 B_2^t \\ 0 & K_e & B_1^t \\ -B_2 C_2^t & B_1 & 0 \end{bmatrix} \begin{bmatrix} u_2 \\ u_e \\ J_{12} \end{bmatrix} = \begin{bmatrix} J_2 \\ J_e \\ 0 \end{bmatrix}$$

FE MESH CONTROL (III)

K_2 stiffness matrix (n_{p2},n_{p2}) of domain Ω_2
C_2 connectivity matrix (n_{p2},n_{T2}) relating the n_{T2} boundary nodes of Ω_2
to the n_{p2} nodes of Ω_2
B_2 matching matrix (n_T,n_{T2}) relating the n_{T2} boundary nodes of Ω_2
to the n_T nodes of Γ_{12}
u_2 node potential vector of domain Ω_2
J_2 nodal current density vector of domain Ω_2
K_e stiffness matrix (n_{T1},n_{T1}) of the equivalent domain replacing Ω_1
B_1 matching matrix (n_T,n_{T1}) relating the n_{T1} boundary nodes of Ω_1
to the n_T nodes of Γ_{12}
u_e node potential vector at the boundary of Ω_1
when Ω_1 and Ω_2 are connected
J_e nodal current density vector of the equivalent domain replacing Ω_1
J_{12} nodal current density vector at Γ_{12} when Ω_1 and Ω_2 are connected

FE MESH CONTROL (IV)

If $n_{T1} = n_T$ and $n_{T_2} \neq n_T$, then matrix B_1 is nothing but the identity of order n_{T1} . As a consequence, the governing equation of vector potential u_2 in domain Ω_2 can be deduced in a compact form:

modified core equation for field analysis in Ω_2

$$\left[K_2 + C_2 B_2^t K_e B_2 C_2^t\right] u_2 = J_2 + C_2 B_2^t J_e$$

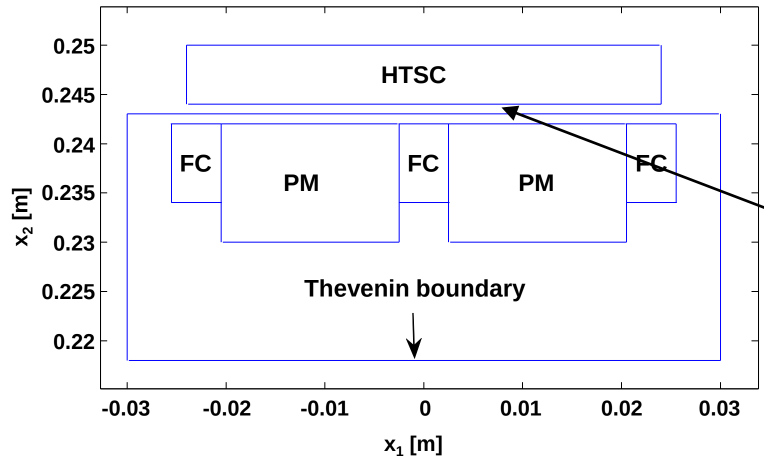
augmented stiffness matrix

→ loss of sparsity

→ condition number higher than that of K_2

CASE STUDY: A *MAGLEV* DEVICE

Device geometry and materials (NdFeB magnet PM, ferromagnetic field corrector FC, high-temperature supercon HTSC)

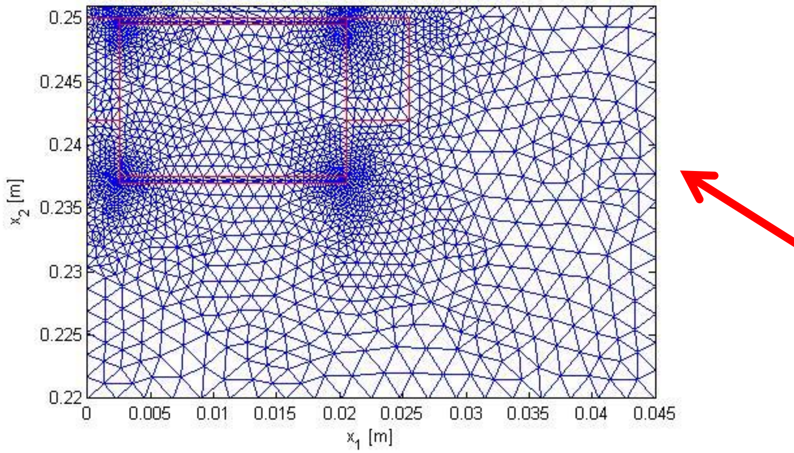


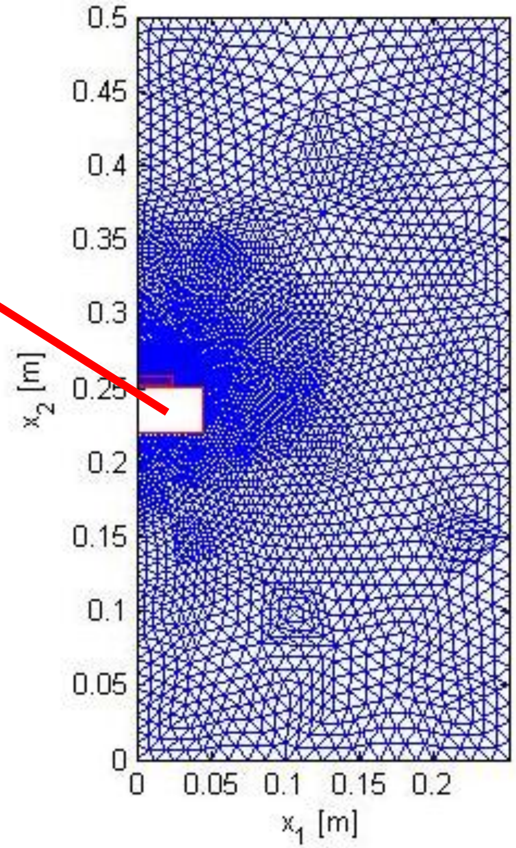
2 mm wide air gap

The HTS sample (48 mm times 6 mm) is supposed to be in the zero-field state; accordingly, it is modelled as a perfectly diamagnetic material.

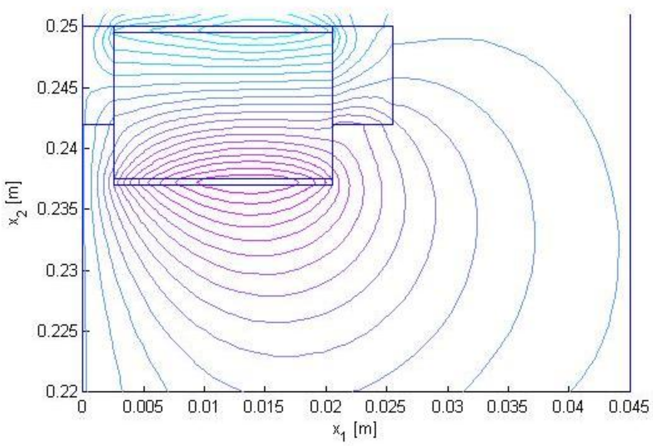
Field diakoptics:
incorporate the excitation system of the *maglev* device (i.e. PMs and FCs) in a subregion bounded by a closed line, along which Thévenin-like conditions in terms of potential are applied.

FIELD ANALYSIS





Reduced domain Ω_2 :
FE mesh (2,014 nodes, 93 Thévenin nodes)
and flux lines



Complementary domain Ω_1 :
FE mesh (5,617 nodes,
129 Thévenin nodes)

FIELD ANALYSIS (II)

In principle, the analysis of the magnetic field in Ω_2 is based on the Poisson's equation in terms of vector potential u, subject to non-homogeneous Dirichlet's condition

$$u = \psi(x_1, x_2) \qquad \text{along } \Gamma_{12}$$

as well as symmetry condition u = 0 along $x_1 = 0$

In the FE scheme, the nodal approximation to function ψ is given by vector u_e , which has two contributions:

$$u_e = K_e^{-1} J_e - K_e^{-1} J_{12} = u_e\big|_{J_{12}=0} + u_{12}$$

no-load potential
(Ω_1 and Ω_2 not connected)

on-load potential
(Ω_1 and Ω_2 connected)

FIELD ANALYSIS (III)

In practice, the analysis of the magnetic field in Ω_2 is based on the direct solution of the FE equation

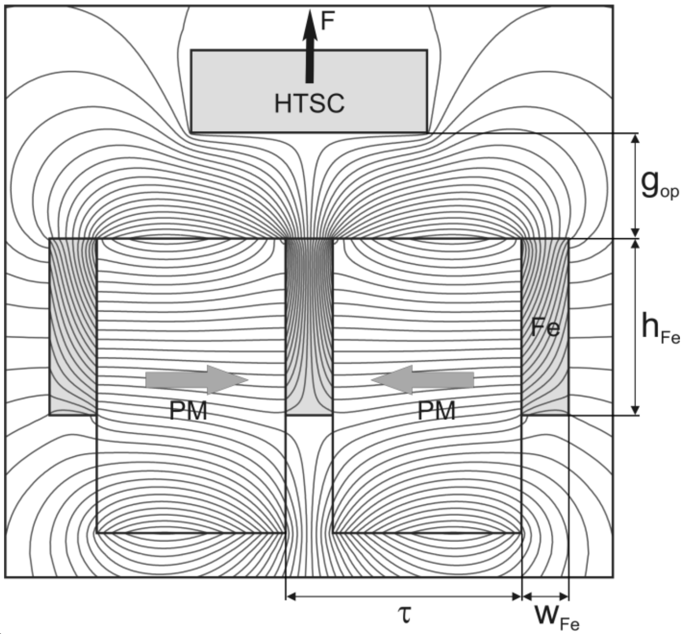
$$\left[K_2 + C_2 B_2^t K_e B_2 C_2^t\right] u_2 = J_2 + C_2 B_2^t J_e$$

in terms of nodal vector potential u_2 , subject to homogeneous Dirichlet's condition $u_2 = 0$ at $x_1 = 0$ (symmetry axis).

FIELD ANALYSIS (IV)

As a result of the interaction between HTS and magnets, a repulsion effect is originated. Elementary levitation force density:

$$dF_2 = JB_1 dS = J\,D_2 u_2\,dS$$



specific current carried by the sheet modelling the magnet

derivative component D_2 acts on the elementary shape function interpolating potential u_2

FIELD SYNTHESIS

Given a sample of HTS, which is assumed to be in the zero-field state, find the magnetic field distribution such that the levitation force density, i.e. the force density F_2 acting on the HTS in the x_2 direction, is maximised.

In practice, the field synthesis problem is converted into a problem of optimal shape design, in which the dimensions of permanent magnets and field correctors are selected as design variables (six-dimensional design vector).

OPTIMISATION PROBLEM

Design vector $\xi \in \Re^6$

ξ_1 and ξ_5 half-width and height of the central field corrector
ξ_2 and ξ_6 width and height of the magnets
ξ_3 and ξ_4 width and height of the lateral field corrector

subject to suitable bounds

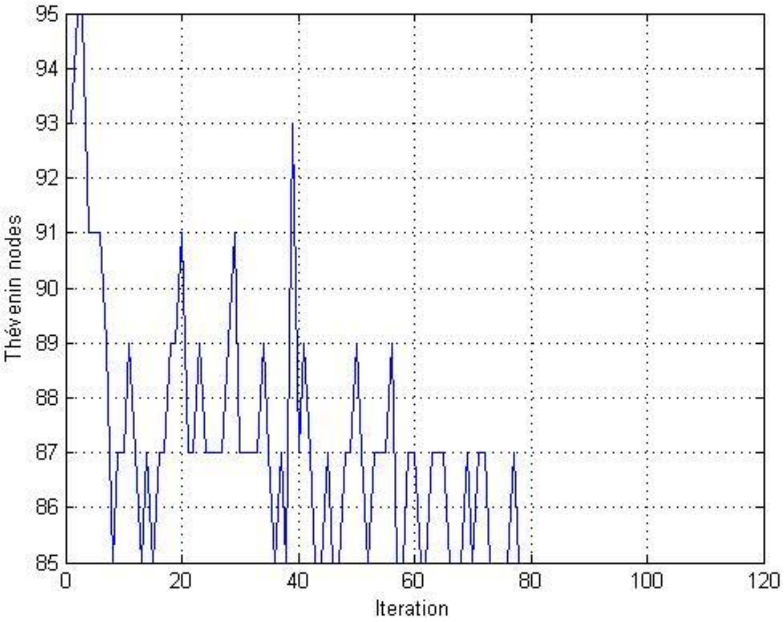
Objective function $$f(\xi) = |F_2(x_2, \xi)|$$

An evolution strategy of lowest order, which has proven to be cost-effective and global-optimum oriented, was used as the optimisation algorithm: a (1+1) scheme, in which the fittest individual between parent and offspring survives to the next generation.

levitation force density acting on the HTSC, to be maximised wrt ξ

OPTIMISATION RESULTS

In a typical optimisation run, the evolution of the mesh discretizing domain Ω_2 makes the number of nodes along the Thévenin boundary Γ_{12} oscillate.

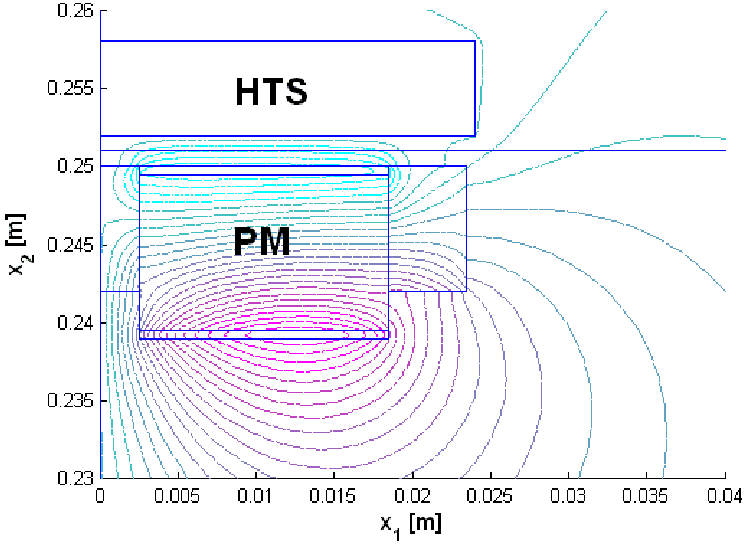


History of the number of Thévenin nodes during the maximisation of the objective f (levitation force density acting on HTSC).

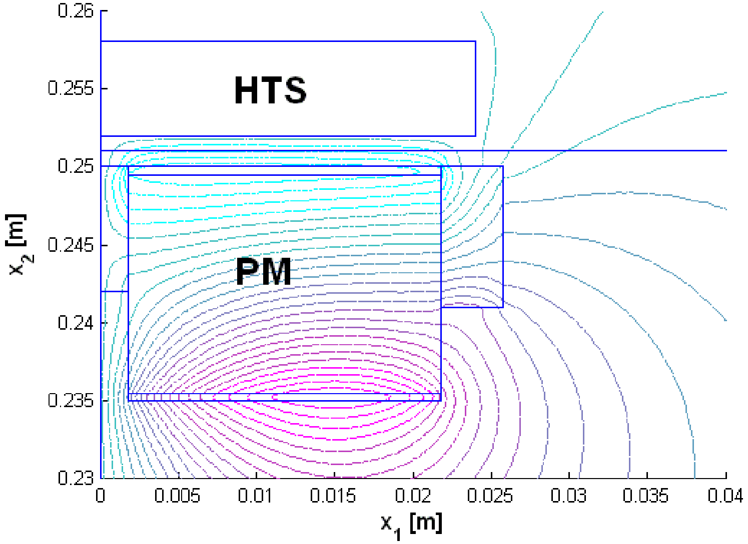
OPTIMISATION RESULTS (II)

Discrete-valued design variables (step equal to 1 mm).

Search tolerance at convergence: $\|\Delta\xi\| / \|\xi\| = 10^{-4}$



Initial geometry with flux lines
F_2 = 1.833 Nmm^{-1}



Final geometry with flux lines
F_2 = 3.446 Nmm^{-1}

The maximisation of levitation force density implies to increase the permanent magnet size and modify the field corrector shape.

CONCLUSION

Thévenin-like conditions are a useful tool to reduce the field domain when field analysis must be repeated, as in the case of synthesis problems.

The example examined refers to magnetic field synthesis in an open-boundary domain.

The reduced field domain can be both linear or non-linear, while the complementary domain must be linear.

A still critical aspect is the construction of connectivity matrices adjusting the nodes of subdomain to the Thévenin boundary.

Suitable meshing techniques, in fact, would help improve the solution of the synthesis problem.